What is a braidoid?

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1 An Overview

Knotoids, introduced by Turaev [16], are immersions of the oriented unit interval into oriented surfaces, endowed with over/under crossing information at each transversal double point. In other words, knotoid diagrams are open-ended knot diagrams as exemplified in Figure 1.

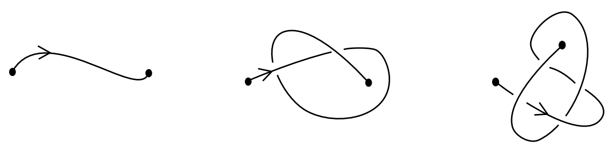

Figure 1: Knotoid diagrams.

A knotoid diagram in S^2 generalizes the notion of a long knot with two endpoints (called *leg* and *head*) that can lie in any planar region complementary to the knotoid diagram. Knotoid diagrams are considered up to three Reidemeister moves, see Figure 2, and the isotopy of the surface they lie in. Moving the endpoints of a knotoid diagram is restricted: We forbid the two moves shown in Figure 3. Precisely, it is forbidden to pull/push an endpoint over/under a strand.

The notion of knotoid can be extended to the notion of multi-knotoid. A *multi-knotoid diagram* in an oriented surface Σ is a union of a knotoid diagram and a finite number of oriented knot diagrams in Σ. Multi-knotoid diagrams in Σ are subject to the Reidemeister moves and isotopy of Σ. The induced isotopy classes of multi-knotoid diagrams are called *multi-knotoids*.

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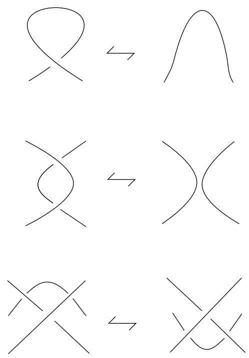

Figure 2: Reidemeister moves

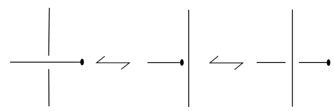

Figure 3: Forbidden knotoid moves.

Knotoids in $\mathbb{R}^2$ admit a 3-dimensional interpretation through planar projections of open-ended curves in 3-dimensional space whose ends are attached on two parallel lines [3], see Figure 4. With this interpretation, knotoids provide a direct and realistic insight for the study of certain entangled physical objects such as proteins [8, 9, 2]. (The reader is referred to the Chapter 36 for further details.)

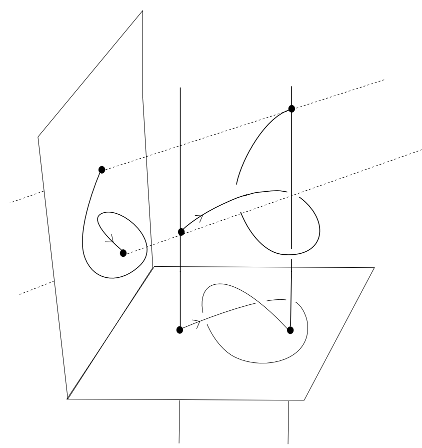

Figure 4: A space curve and its different planar projections as knotoids.

Here we present *braidoids* introduced by the author and Lambropoulou [6, 7]. A braidoid generalizes the notion of a classical braid: It is a system of descending strands interacting with each other at a finite number of points assigned with a specified over and under information. The ends of strands are assumed to be fixed at top and bottom except the two ends that are not necessarily lying at top or bottom and are subject to special topological moves that we explain below. Before going into the precise definition let us view some examples of braidoid diagrams given in Figure 5.

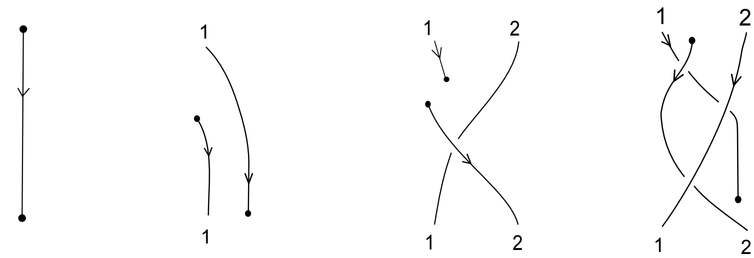

Figure 5: Some examples of braidoid diagrams.

2 The Precise Definition of a Braidoid

2.1 Braidoid Diagrams

Let I denote the unit interval $[0, 1] \subset \mathbb{R}$. We identify $\mathbb{R}^2$ with the xt-plane with the t-axis directed downward. A *braidoid diagram* B is a system of a finite number of arcs lying in $I \times I \subset \mathbb{R}^2$. The arcs are called the *strands* of B. Each strand is naturally oriented downward with no local maxima or minima, and there is only a finite number of intersection points among the strands which are transversal double points endowed with over/under data. Such intersection points are called *crossings* of B.

B has two types of strands. A *classical strand* is like a braid strand connecting two points, one that lies on $I \times \{0\}$ and the other lies on $I \times \{1\}$. Each end of a classical strand is assumed to be fixed. The other type of strands, the so called *free strands*, are of a more flexible nature: One or two ends of a free strand are located at points which are not necessarily on $I \times \{0\}$ or on $I \times \{1\}$, and they are not assumed to be fixed. Such ends of free strands are denoted by graphical nodes to be distinguished from the fixed ends (that might be the ends of either classical or free strands), and are called *endpoints* of B. B has exactly two endpoints. We call an endpoint a *head* if it is the terminal point of a free strand and a *leg* if it is the beginning point of a free strand, in analogy with the leg and the head of a knotoid diagram. Each fixed end of B lying on $I \times \{0\}$ is paired up with a fixed end on $I \times \{1\}$ that is in the same vertical alignment with it. Paired ends are called *corresponding ends.* Each pair of corresponding ends are numbered from left to right, as we show in Figure 5.

2.2 Isotopy Moves of Braidoid Diagrams

We allow Δ-*moves* on braidoid diagrams that replace a segment of a strand with two segments in a triangular region free of any of the endpoints (see Figure 6).

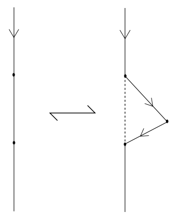

Figure 6: A Δ-move.

A Δ-move on a braidoid diagram preserves the downward orientation of the braidoid strands and respects the crossing information of the strands intersecting the triangular region of the move. The Reidemeister II and III moves in which the downward direction of strands is preserved, are special cases of the Δ-moves on braidoid diagrams. Similar to the forbidden moves of knotoid diagrams, moving the endpoints of a braidoid diagram over/under a strand, as shown in Figure 7, are *forbidden moves* for braidoid diagrams. It is clear that allowing forbidden moves would cancel any braiding of the free strands.

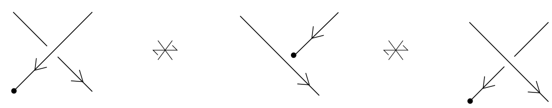

Figure 7: Forbidden braidoid moves.

The following moves displacing the endpoints are allowed on braidoid diagrams.

1. *Vertical moves*: An endpoint can be pulled up/down along the vertical direction as long as the forbidden moves are not violated (see Figure 8).

2. *Swing moves*: An endpoint can be moved to the right and to the left like a pendulum (see Figure 9), as long as the downward orientation on the swinging arc is preserved and the forbidden moves are not violated.

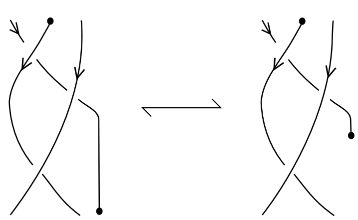

Figure 8: A vertical move.

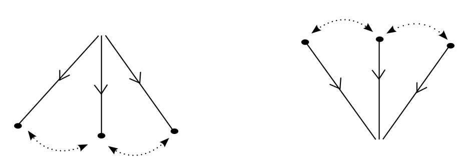

Figure 9: Swing moves.

Definition 1 A *braidoid* is an isotopy class of braidoid diagrams taken up to the isotopy relation generated by the oriented Reidemeister II and III moves and planar Δ moves together with the swing and vertical moves for the endpoints.

3 Braidoids in Relation with Knotoids

In 1923 Alexander showed that any classical knot and link can be represented in braided form [1], then in 1936 Markov proved a one-to-one correspondence between the set of classical links considered up to link isotopy and the set of braids considered up to a braid equivalence relation generated by braid isotopy moves, conjugation and stabilization [15]. We have analogues of Alexander's and Markov's theorems that relate braidoids equipped with a special labeling to multi-knotoids in $\mathbb{R}^2$.

3.1 Labeled Braidoid Diagrams

Definition 2 A *labeled braidoid diagram* is a braidoid diagram such that each pair of its corresponding ends —not the endpoints—is labeled either with o or u.

Labeled braidoid diagrams are subject to the braidoid Δ-moves, the vertical moves, and the *restricted swing moves*, shown in Figure 10, whereby the swinging of an endpoint takes place within the interior of the vertical strip determined by the vertical lines that pass through two neighboring pairs of corresponding ends. During the restricted swing moves, the endpoints cannot surpass the vertical line determined by any pair of corresponding ends. Clearly, a restricted swing move is transformed into a planar isotopy move by the closure defined below. The reason for restricting the swing moves for labeled braidoid diagrams is to avoid any incident of forbidden moves on the multi-knotoid diagram obtained by the closure. See Figure 13 that depicts how an unrestricted swing move gives rise to nonequivalent knotoids under the closure.

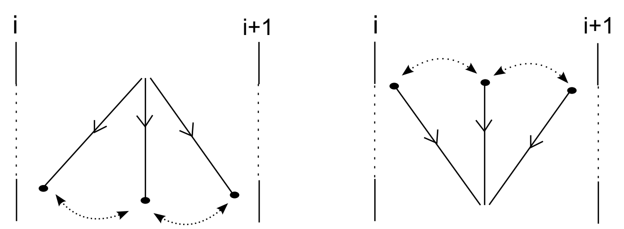

Figure 10: Restricted swing moves.

Definition 3 *Labeled braidoid isotopy* is generated by the braidoid Δ-moves, vertical moves and the restricted swing moves, preserving at the same time the labeling. Equivalence classes of labeled braidoid diagrams under this isotopy relation are called *labeled braidoids*.

3.2 A Closure for Labeled Braidoids

The correspondence between labeled braidoid diagrams and multi-knotoid diagrams in $\mathbb{R}^2$ is based on a closure operation that is defined in analogy with the closure of braid diagrams in handlebodies [13] (see the Chapter 7 for further details).

In this closure operation, each pair of the corresponding ends of a labeled braidoid diagram is connected with an arc that goes entirely over or under the rest of the diagram according to the label of ends, as illustrated in Figure 11 and in Figure 12.

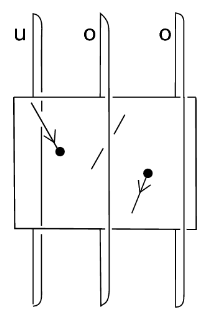

Figure 11: Closure of a labeled braidoid diagram.

Precisely, if a pair is labeled with o then the connecting arc goes entirely over the diagram and if a pair is labeled with u then the connecting arc goes entirely under the diagram. The connecting arcs lie on the right-hand side of the pairs of corresponding ends, in an arbitrarily close distance to the vertical lines determined by these pairs so that there is no endpoint of the braidoid diagram between the vertical lines and the connecting arcs. The resulting diagram is clearly a knotoid or a multi-knotoid diagram in the plane. Note that the closure of the trivial braidoid diagram shown in Figure 5 that has no pairs of corresponding braidoid ends, is assumed to be the trivial knotoid. Also, the isotopy class of the resulting multi-knotoid diagram depends on the labeling. A braidoid diagram equipped with two different labelings might yield to non-isotopic closures, as shown in Figure 12 (it is left as an exercise for the interested reader to verify the knotoid on the right-hand side of the figure is actually non-trivial).

It is shown in [6, 7] that the closure operation induces a well-defined map on the set of labeled braidoids to the set of multi-knotoids in $\mathbb{R}^2$.

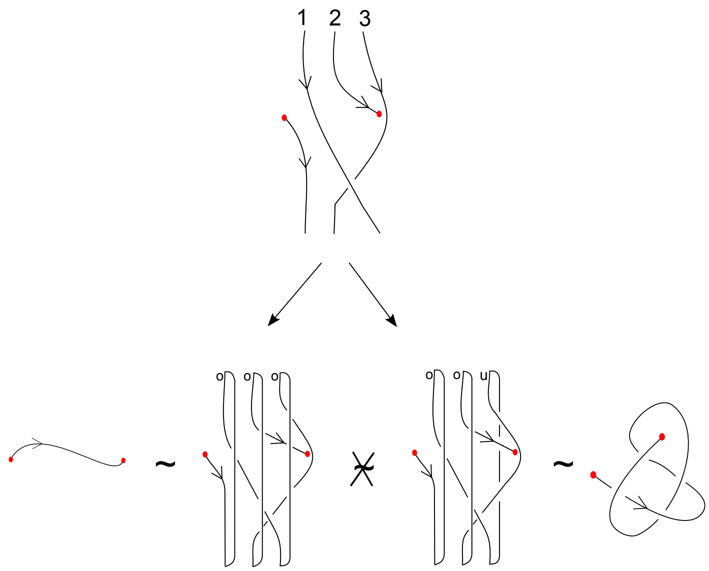

Figure 12: Two different labelings and nonequivalent knotoids upon closure.

Figure 13 illustrates an unrestricted swing move on an endpoint of a labeled braidoid diagram that corresponds to a knotoid forbidden move upon closure.

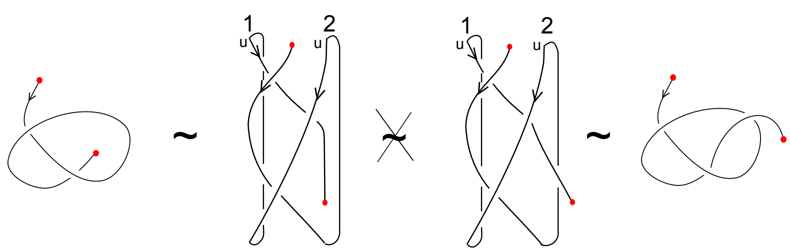

Figure 13: Swing moves may give rise to nonequivalent knotoids.

3.3 How to Turn a Knotoid Diagram into a Braidoid Diagram

Let K be a knotoid or a multi-knotoid diagram in a plane equipped with the top-to-bottom direction. One way to obtain an inverse operation to the closure we define above, that is, to transform K to a labeled braidoid diagram whose closure is isotopic to K, is to manipulate K by eliminating its arcs oriented upward, namely the *up-arcs*. Before the elimination begins, the knotoid diagram K is subdivided (marked with dots) starting from its local maxima and minima until each of its up-arcs contains only one type of crossings (either over- or under-crossings) or contains no crossings at all. We label each up-arc with o or u according to the type of crossings it contains. If an up-arc does not contain any crossings, then we are free to label the up-arc either with o or u. Braidoiding moves are analogues of the braiding moves defined for classical braids [11, 12, 14].

Definition 4 A *braidoiding move* consists of cutting an up-arc at a point (we call a *cut-point*), and then pulling the resulting ends entirely over or under the rest of the diagram according to the label of the up-arc, to $I \times \{0\}$ and $I \times \{1\}$ preserving the alignment with the cut-point (see Figure 14). A cut-point is chosen so that it is not vertically aligned with another cut-point or with an endpoint of K. To obtain a pair of braidoid strands (that is, monotonically descending strands with a label), we complete the braidoiding move by Δ-moves applied to the upward-directed pieces of the resulting strands (see the second step shown in Figure 14). Finally, we label the resulting braidoid strands with o or u with respect to the label of the up-arc eliminated. Notice that in the last instance of Figure 14, the pair of the resulting corresponding ends is joined up together with a connecting arc that goes entirely over the diagram in accord with the labeling. By this, we obtain a closed strand which can be retracted back to the initial up-arc QP. Note that during this isotopy, a violation of the knotoid forbidden moves is avoided by choosing the connecting arc close enough to the vertical line determined by the pair of corresponding ends.

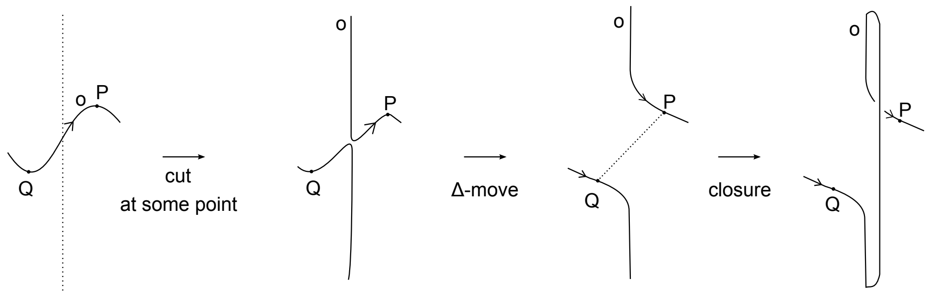

Figure 14: A braidoiding move and its closure.

In [6, 7] we present two *braidoiding algorithms* for turning a planar multi-knotoid diagram into a braidoid diagram. Both of these algorithms are based on the *braidoiding moves*. Figure 15 exhibits one of the braidoiding algorithms which consists of rotating a crossing of a given multi-knotoid diagram by 90 degrees if the crossing is contained in one up-arc and by 180 degrees if the crossing is contained in two up-arcs and then applying the braidoiding moves on the resulting multi-knotoid diagram. The reader is encouraged to see the closure of the labeled braidoid diagram obtained in Figure 15 is isotopic to the given knotoid diagram, and is referred to [6, 7] for the technical details of the algorithms.

The closure defined for labeled braidoid diagrams and the braidoiding algorithms provide us the following theorem [6, 7, 5] that is an analogue of Alexander's theorem [1].

Theorem 5 *Any multi-knotoid diagram in $\mathbb{R}^2$ is isotopic to the closure of a labeled braidoid diagram.*

The braidoiding algorithm presented in Figure 15 also provides the following sharpened version of Theorem 5 [6, 5].

Theorem 6 *Any multi-knotoid diagram in $\mathbb{R}^2$ is isotopic to the closure of a labeled braidoid diagram whose corresponding ends labeled only with u.*

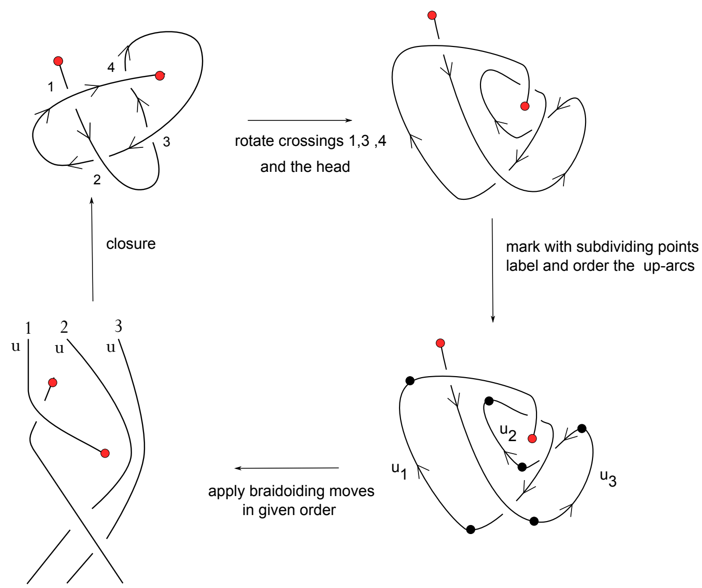

Figure 15: A knotoid diagram and the associated labeled braidoid diagram.

3.4 A Geometric Markov Theorem for Braidoids

It is possible that two non-isotopic labeled braidoid diagrams in a plane close to isotopic planar knotoids or multi-knotoids (consider the closures of the first three braidoid diagrams in Figure 5 with some labeling). A natural question arising is the following: *Is there a way to define a relation between labeled braidoids that have isotopic closures?*

In [7] we give a geometric relation between labeled braidoid diagrams closing to isotopic knotoids or multi-knotoids. This relation is induced by the L-moves [6, 7] and some special swing moves defined on labeled braidoid diagrams. The L-moves were originally defined for classical braids by Lambropoulou [11, 12, 14], and utilized to prove a one-move Markov theorem for classical braids [11].

There are two types of L-moves as shown in Figure 16. An L_o- move (respectively an L_u-move) consists of cutting a labeled braidoid strand at an interior point and pulling the ends of the resulting sub-strands to the top and bottom lines over the rest of the diagram (respectively under the rest of the diagram) so that a new pair of strands is obtained whose ends are vertically aligned with the cut-point. The resulting strands are labeled with o (respectively with u). Notice that when the corresponding ends of the resulting strands are connected with an overpassing arc (respectively with an underpassing arc), we obtain a closed strand which is isotopic to the original braidoid strand. This observation is sufficient to deduce that two labeled braidoid diagrams which are related to each other via L-moves have isotopic closures.

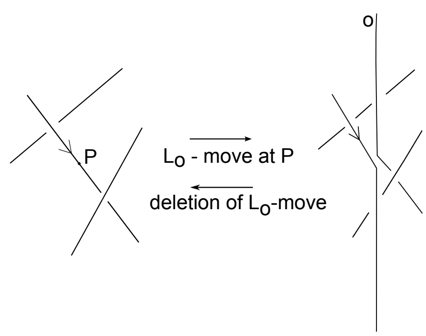

Figure 16: An L_o-move on a braidoid strand.

We also observe that there are some type of unrestricted swing moves that yield isotopic knotoids or multi-knotoids through the closure. With this type of swing moves, the endpoint surpasses the vertical line determined by a pair of corresponding ends but this do not affect the isotopy class of the resulting (multi-)knotoid diagram. See Figure 17. We call these moves *fake swing moves*.

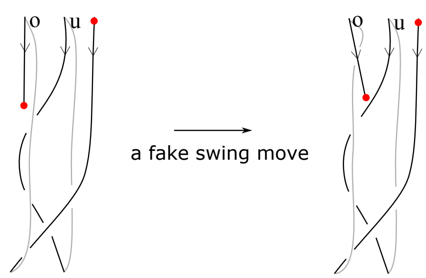

Figure 17: A fake swing move.

The L-moves together with the fake swing moves extend the labeled braidoid isotopy and provide us the following theorem that is an analogue of Markov's theorem [15] for labeled braidoids.

Theorem 7 *The closures of two labeled braidoid diagrams are isotopic multi-knotoid diagrams in $\mathbb{R}^2$ if and only if the labeled braidoid diagrams relate to* *each other by a sequence of labeled braidoid isotopy moves, addition/deletion of L-moves, and fake swing moves.*

For proving this theorem, we examine the effect of all possible algorithmic choices made in the braidoiding algorithms and the transformation of the knotoid isotopy moves under the braidoiding moves. The reader is referred to [6] for the details of the proof.

4 Discussion

Braidoids provide a new diagrammatic theory that is, in fact, a braid theory for the theory of knotoids. An underlying combinatorial structure for braidoids is discussed in [5, 7], In this combinatorial structure, braidoid diagrams can be partitioned into a finite number of *building blocks* extending the braiding generators. With the algebraic expressions corresponding to the building blocks and the Alexander and Markov-type theorems discussed throughout this paper, braidoids suggest an algebraic tabulation for polymers, specifically for proteins [7].

Acknowledgement

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