On Knotoids, Braidoids and Their Applications

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References

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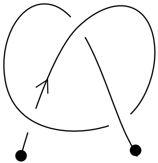
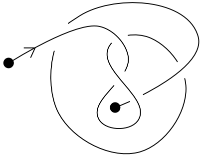
What is a knotoid diagram?





A *knotoid diagram* in S^2 or in $\mathbb{R}^2$ is an open-ended knot diagram with two endpoints that can lie in different regions of the diagram.

What is a knotoid diagram?

Definition (Turaev)

A *knotoid diagram* K in an oriented surface Σ is an immersion

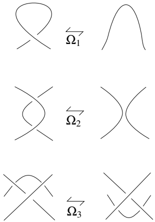
$$K : [0,1] \to \Sigma \text{ such that:}$$

1. each transversal double point is endowed with under/over data, and we call them *crossings* of K,
2. the images of 0 and 1 are two disjoint points regarded as the *endpoints* of K. They are called the *leg* and the *head* of K, respectively.
3. K is oriented from the leg to the head.

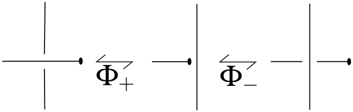
What is a knotoid?

Definition

A *knotoid* in Σ is an equivalence class of the knotoid diagrams in Σ up to the equivalence relation induced by the Ω-moves and isotopy of Σ.



Ω-moves



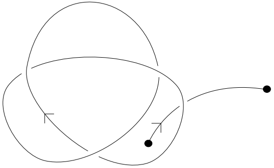
Forbidden knotoid moves

Extending the definition of a knotoid

Definition (Turaev)

A *multi-knotoid diagram* in an oriented surface Σ is a generic immersion of a single oriented segment and a number of oriented circles in Σ endowed with under/over-crossing data.

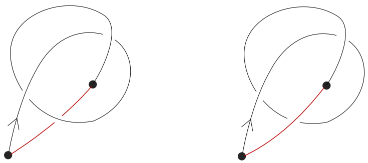
A *multi-knotoid* is an equivalence class of multi-knotoid diagrams determined by the equivalence relation generated by Ω-moves and isotopy of the surface.



A multi-knotoid diagram

We mostly focus on the case $\Sigma = S^2$ or $\mathbb{R}^2$.

From knotoids to classical knots



There is a surjective map,

$$\omega_{-}: \{ \text{ Knotoids } \} \to \{ \text{ Classical knots } \}$$

induced by connecting the endpoints of a knotoid diagram with an underpassing arc.

$\Rightarrow$ Invariants of classical knots can be computed on knotoid representatives.

- Let κ be a knot and K be a knotoid representative of κ. Then $\pi_1(\kappa) = \pi(K)$ (Turaev).
- A knot is tricolorable if and only if it admits a tricolorable knotoid representative.

What is an invariant of a knotoid?

Definition

Let M denote a set of mathematical objects. An *invariant of knotoids* is a mapping

$$I : \{\text{Knotoids}\} \to M,$$

assigning to equivalent knotoid diagrams the same value.

Some of the former knotoid invariants given by Turaev:

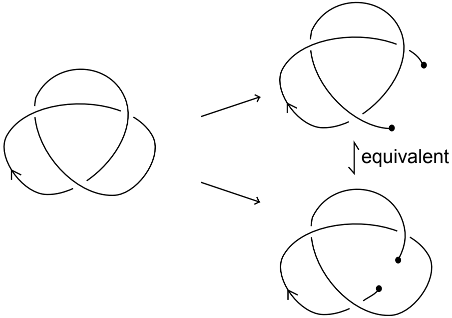
- The bracket polynomial
- 2-variable extended bracket polynomial: Knotoids in S^2 have been classified up to 5 crossings, by Bartholomew using this invariant.
- 3-variable bracket polynomial: We have used this polynomial for our analysis of protein chains via planar kntooids.

From classical knots to knotoids

- There is an injective map,

$$\alpha\colon \ \{\text{Classical knots}\} \to \{\text{Knotoids in } S^2\},$$

induced by deleting an open arc which does not contain any crossings from an oriented classical knot diagram.



$\Rightarrow$ The theory of knotoids in S^2 is an extension of classical knot theory.

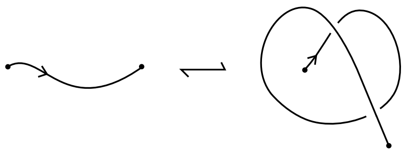
From classical knots to knotoids

Definition

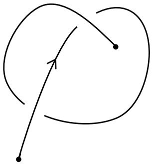
A knotoid in S^2 that is in the image of α, is called a *knot-type* knotoid.
A knotoid that is not in the image of α, is called a *proper* knotoid.

$$\{\text{Knotoids in } S^2\} = \{\text{Knot-type knotoids}\} \cup \{\text{ Proper knotoids}\}$$

Knot-type knotoids carry the same topological information with corresponding classical knots.



A knot-type knotoid



A proper knotoid

The height of a knotoid

Definition (Turaev)

The *height* of a knotoid diagram is the minimum number of crossings created by the underpass closure.
The *height of a knotoid K* is defined as the minimum of the heights, taken over all equivalent knotoid diagrams to K.

The height is a knotoid invariant.

A knotoid has zero height if and only if it is a knot-type knotoid.

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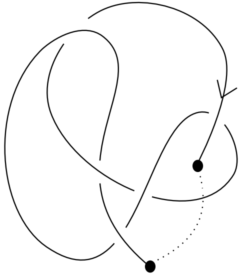
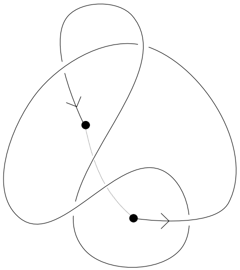
A knotoid has non-zero height if and only if it is a proper knotoid.

How do we compute the height?





Question

Apparently, the first diagram has height 1 and the second one has height 2. But are there some equivalent diagrams to the knotoids above with less height?

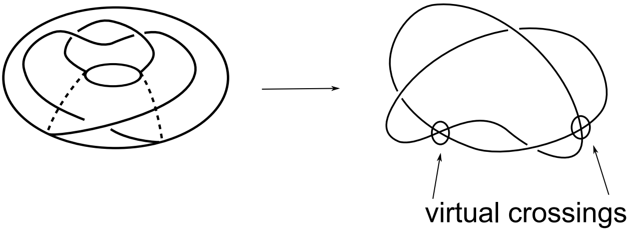
Virtual knots

Definition (Kauffman)

A *virtual knot* k is an embedding,

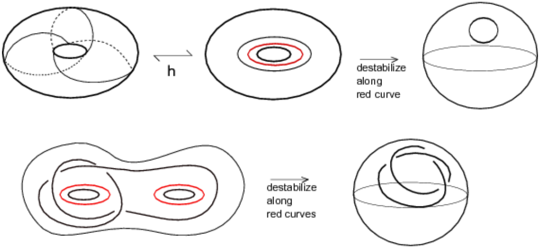
$$k : S^1 \to \Sigma_{g \geq 0} \times [0,1].$$

A *virtual knot diagram* is a generic projection of k in Σ_g with the data of under/over assigned to each self-crossings.



Definition (Stable-equivalence)

The virtual knot diagrams in two surfaces are said to be *stably-equivalent* if one is related to the other one by the three Reidemeister moves in the surfaces, orientation preserving homeomorphisms of the surfaces and addition/removal of 1- handles in the complements of the diagrams.



Definition

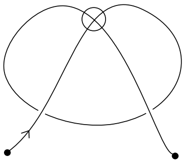
The *genus* of a virtual knot is the minimum genus among the surfaces that the knot has a diagram without any virtual crossings.

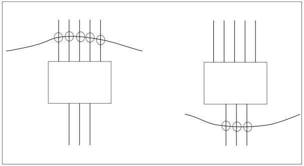
Virtual knotoids

The notions of virtual knot theory naturally extends to knotoids.

Definition

A *virtual knotoid diagram* is a knotoid diagram in S^2 (or in $\mathbb{R}^2$) with classical and virtual crossings.
A *virtual knot* is an equivalence class of virtual knotoid diagrams up to the equivalence relation generated by the Ω-moves and the detour move.





The following theorem was stated by Turaev.

Theorem (G.,Kauffman)

The theory of virtual knotoids is equivalent to the theory of knotoid diagrams in higher genus surfaces considered up to Ω*-moves in the surfaces, isotopy of the surfaces and addition/removal of handles in the complement of knotoid diagrams.*

From knotoids to virtual knots

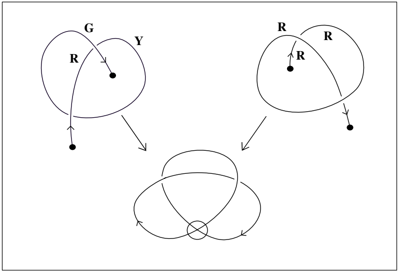
There is a well-defined map called the *virtual closure map*,

$$\overline{v}\colon \{\text{Knotoids in } S^2\} \to \{\text{Virtual knots of genus} \leq 1\ \},$$

induced by associating a knotoid diagram to the virtual knot diagram obtained by connecting the endpoints of the knotoid diagram in the virtual fashion.

The virtual closure map

- The virtual closure map is not injective:



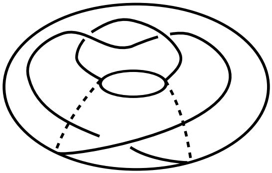
A pair of nonequivalent knotoids with the same virtual closure

The virtual closure map

Proposition (G.,Kauffman)

The virtual closure map is not surjective.

The virtual knot below is a genus 1 knot that is not in the image of $\overline{v}$.



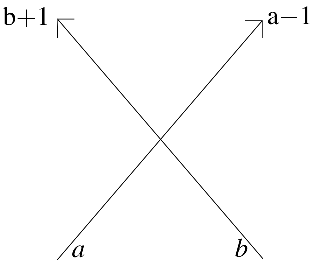
- We showed this by examining the surface-state curves of the diagram in torus.

Yet, since the virtual closure map is well-defined, any virtual knot invariant can be defined as an invariant of knotoids through this map.

- Let *Inv* denote an invariant of virtual knots.
  Define an invariant of knotoids I by the following formula,
  $$I(K) = Inv(\bar{v}(K)),$$
  where K denotes a knotoid in S^2.
- A second notice shows that many virtual knot invariants can be directly constructed on knotoids.

The affine index polynomial of knotoids

We label each *arc* of the flat diagram of a knotoid diagram K by an integer with respect to the following rule:



Integer labeling at a flat crossing

Note that the first and the last arc of the diagram can be always labeled by 0.

The affine index polynomial of a knotoid

Definition

Let c be a crossing of K. Two number outcomes $w_+(c)$ and $w_-(c)$ are derived at c which are called *positive* and *negative* weights of c:

$$w_+(c) = a - (b+1)$$
$$w_-(c) = b - (a-1)$$

where a and b are the labels for the left and the right incoming arcs at the corresponding flat crossing to c, respectively.
The *weight* of c is defined as

$$w_K(c) = \begin{cases} w_+(c) & \text{if the sign of } c \text{ is positive,} \\ w_-(c) & \text{if the sign of } c \text{ is negative} \end{cases}$$

The affine index polynomial of a knotoid

Definition (G., Kauffman)

The *affine index polynomial* of a knotoid K is defined by the equation,

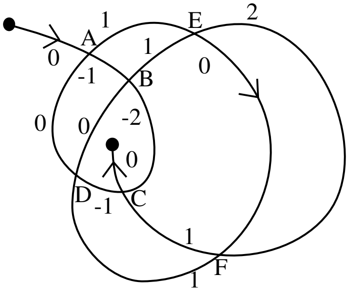
$$P_K(t) = \sum_c \operatorname{sign}(c)(t^{w_K(c)} - 1)$$

where the sum is taken over all crossings of a diagram of K.

The affine index polynomial is a knotoid invariant.

- The affine index polynomial of a knot-type knotoid is trivial. This follows by the Jordan curve theorem.
- Proper knotoids may have non-trivial affine index polynomial.

<u>An example:</u>



	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1

$$P_K(t) = t^2 + 2t + 2t^{-1} + t^{-2} - 6$$

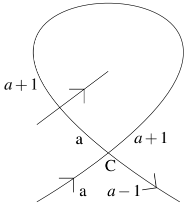
The affine index polynomial and the height

Theorem (G.,Kauffman)

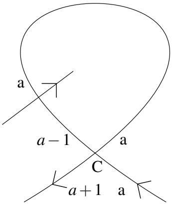
The height of a knotoid is greater than or equal to the maximum degree of its affine index polynomial.

A sketch of the proof

Main Observation: Any crossing determines a loop starting and ending at that crossing. The algebraic intersection number of the loop at c with the intersecting arcs is equal to either $w_-(c)$ or $w_+(c)$.



Entering Value $-$ Exit Value $= w_-(C)$

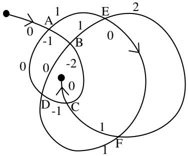


Entering Value $-$ Exit Value $= w_+(C)$

The loop at a maximal weight crossing reveals information about the height of the diagram when self-intersections on the loop are smoothed in the oriented way.

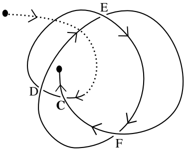
An illustration

An illustration:

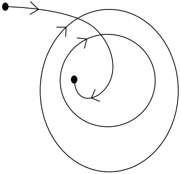


	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1

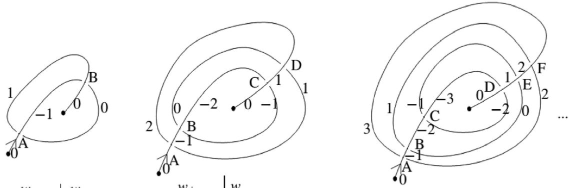
$$P_K(t) = t^2 + 2t + 2t^{-1} + t^{-2} - 6$$



The loop at C



Seifert circles after smoothing



	w_+	w_-
A	-1	$+1$
B	$+1$	-1

	w_+	w_-
A	-2	$+2$
B	-1	$+1$
C	$+2$	-2
D	$+1$	-1

	w_+	w_-
A	-3	+3
B	-2	+2
C	-1	+1
D	+3	-3
E	+2	-2
F	+1	-1

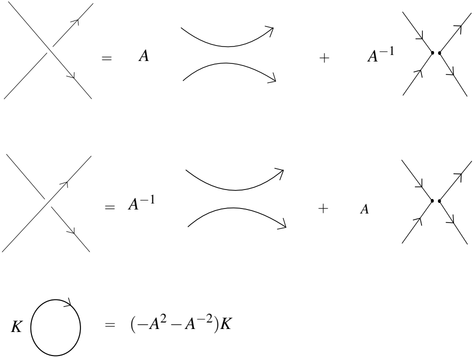
- $P(K_1) = t + t^{-1} - 2.$
- $P(K_2) = t^2 + t + t^{-1} + t^{-2} - 4.$
- $P(K_3) = t^3 + t^2 + t + t^{-1} + t^{-2} + t^{-3} - 6.$

These are some knotoids whose heights are detected by the affine index polynomial.

We have infinitely many knotoids whose heights are detected by the affine index polynomial.

An extension of the bracket polynomial: The arrow polynomial

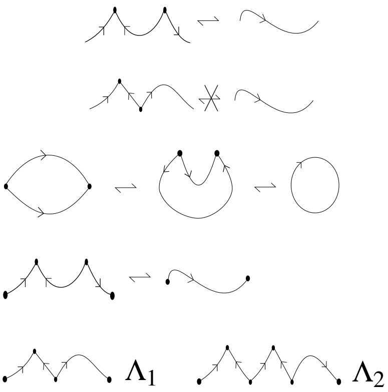
The construction of the arrow polynomial for knotoids is based on the *oriented state expansion* of the bracket polynomial.



Oriented state expansion

Reduction rules for the arrow polynomial

To reduce the number of cusps in a state component we have the following rules:



The arrow polynomial

Definition (G.,Kauffman)

We define the *arrow polynomial* of a knotoid diagram K as,

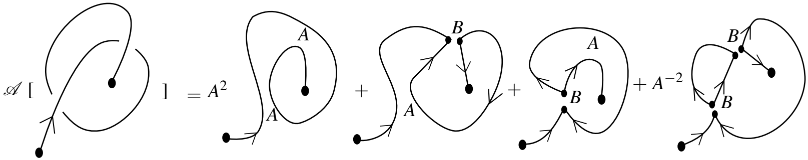
$$\mathscr{A}[K] = \sum_S < K|S > (-A^2 - A^{-2})^{\|S\|-1}\Lambda_i$$

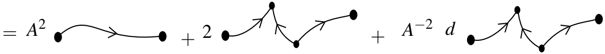
where $< K|S >$ is the usual vertex weights of the bracket polynomial, $\|S\|$ is the number of components of the state S and Λ_i is the variable associated to the long segment component of S with irreducible zig-zags.

Theorem (G.,Kauffman)

The normalization of the arrow polynomial (multiplication by $(-A^{-3})^{-\mathrm{wr}(K)}$*) is a knotoid invariant.*

An example for computing the arrow polynomial





$= A^2 + (1 - A^4)\Lambda_1$

The arrow polynomial and the height

Definition

The Λ-degree of a summand of the arrow polynomial of a knotoid which is in the form, $A^m\Lambda_i$ is equal to i. The Λ-*degree* of the arrow polynomial of a knotoid is defined to be the maximum Λ- degree among the Λ-degrees of all the summands of the polynomial.

Lemma (G.,Kauffman)

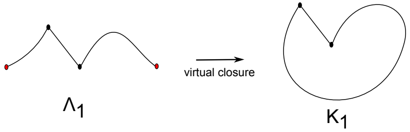
The arrow polynomial of a knot-type knotoid has zero Λ-degree.

Theorem (G.,Kauffman)

The height of a knotoid K in S^2 is greater than or equal to the Λ-degree of its arrow polynomial.

Idea of the proof

The proof is based on the following observations:

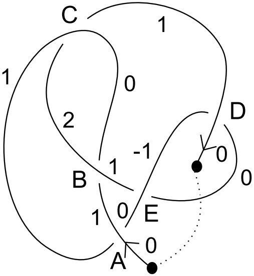


- The height of a knotoid diagram is equal to the virtual crossing number of its virtual closure.
- The relation of the virtual crossing number of virtual knots and the K-degree of the arrow polynomial for virtual knots:

Theorem (Dye,Kauffman)

The minimum number of virtual crossings of a virtual knot is greater than or equal to the K-degree of the arrow polynomial of the knot.

Comparison on the two lower bounds

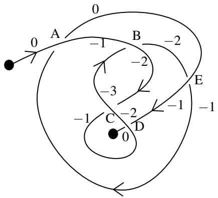


A	0	0
B	-1	1
C	1	-1
D	1	-1
E	-1	1

- $P(K) = (t^0 - 1) - (t - 1) - (t^{-1} - 1) + (t - 1) + (t^{-1} - 1) = 0$
- $\mathscr{A}[K_1] =$ $-A^{-3} + A - 2A^5 + A^9 + \Lambda_1(A^{-9} - 2A^{-5} + 2A^{-1} - 2A^3 + A^7)$.

$\Rightarrow$ The height of the knotoid is 1 by the arrow polynomial.

Further on the height



	w_+	w_-
A	0	0
B	1	-1
C	-1	1
D	1	-1
E	-1	1

- $P_K(t) = 2t + 2t^{-1} - 4$
- $\mathscr{A}[K] = (-A^{-5} + 2A^{-1} - A^3 - A^7) + 2(A - A^5)\Lambda_1.$

$\Rightarrow 1 \leq h(K) \leq 2$. What is the height of K?

Theorem (G.,Kauffman)

Minimal diagrams of knot-type knotoids have zero height.

Conjecture

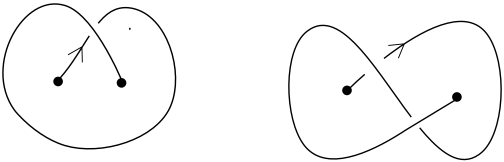
Minimal crossing number of knotoids in S^2 is admitted on minimal height diagrams.

Comparing knotoids in S^2 and $\mathbb{R}^2$

There is a surjective map,

$$\iota: \text{The set of knotoids in } \mathbb{R}^2 \to \text{The set of knotoids in } S^2$$

which is induced by $\iota : \mathbb{R}^2 \hookrightarrow S^2$. This map is not injective.



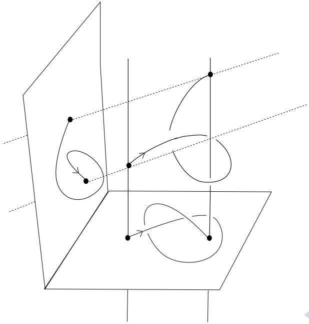
Nontrivial planar knotoids which are trivial in S^2

A geometric interpretation for planar knotoids

Two open space curves with their ends attached on two parallel lines are *line isotopic* if there is an ambient isotopy taking one curve to the other in the complement of the lines and keeping the ends on the lines.

Theorem (G., Kauffman)

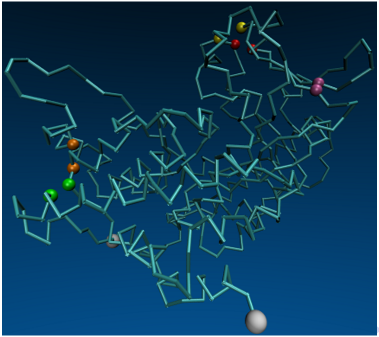
There is a bijection between the set of knotoids in $\mathbb{R}^2$ *and the set of line isotopy classes of smooth open oriented curves in* $\mathbb{R}^3$.



Analyzing the topology of open protein chains

Classical Approaches:

1. **Direct closure**: Connect the two endpoints of a protein chain and analyze the type of the resulting knot.
2. **Uniform closure**: Place the protein chain in a ball
   Choose a point on the boundary of the ball and connect two endpoints of the chain to this point and determine the resulting knot type.



Analyzing open protein chains using knotoids

A new approach: (joint work with Goundaroulis, Dorier, Stasiak, Lambropoulou and Kauffman)

1. The protein chain is assumed to lie in a ball of sufficiently large radius.
2. Each point on the boundary determines a projection plane for the protein chain.
3. Choose one plane and introduce the lines passing through the termini and perpendicular to the plane.
4. Simplify the chain by an algorithm eliminating triangles by never crossing through the lines.
5. Project the chain to the plane along the lines.
6. The resulting diagram is a knotoid diagram. Determine the knotoid type by using knotoid invariants.

Analyzing the protein 3KZN

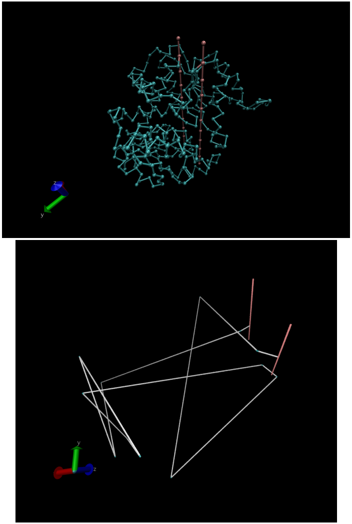
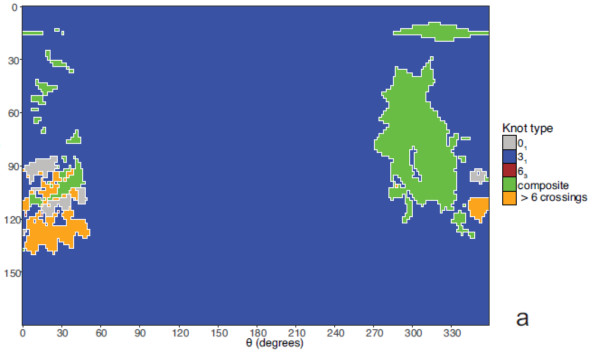
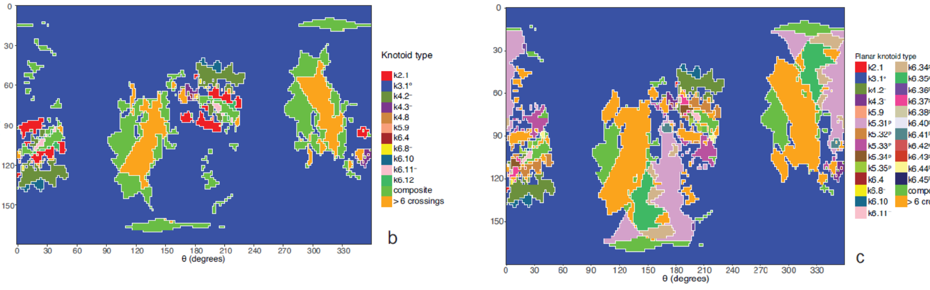


Figure: The protein 3KZN



Detecting 3KZN via uniform closure



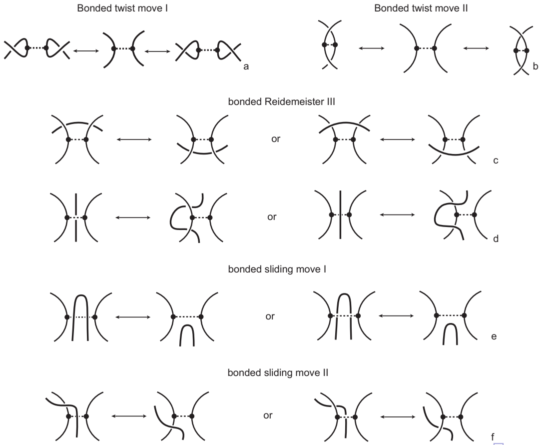
Detecting 3KZN via knotoids, b:spherical knotoids, c: planar knotoids

Conclusion

Analyzing open protein chains as planar knotoids reveals more details of their topology.

A topological model for bonded protein chains

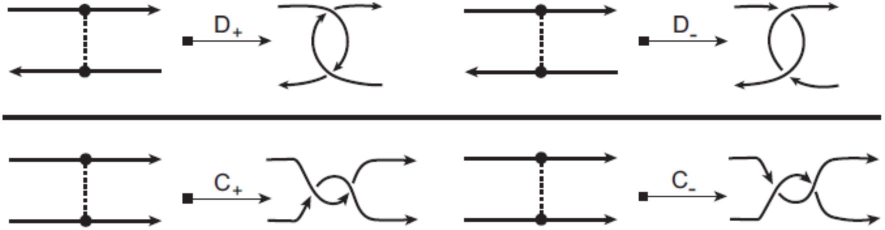
<u>Definition</u>: A *bonded knotoid diagram* is a knotoid diagram with finitely many edges connecting any two strands of the diagram.
A *bonded knotoid* is an equivalence class of bonded knotoid diagrams up to the equivalence relation generated by the bonded moves:



A topological modeling of bonded protein chains

1. Determine a projection direction for the chain with bonds.
2. Draw the two lines passing through the endpoints of the chain and are perpendicular to the projection plane. Simplify the chain accordingly to the lines.
3. Project the chain to the plane along the lines. The bonds of the protein are projected as edges between the corresponding points. The resulting diagram is a bonded knotoid diagram.
4. Replace the bonding site by a full-twist along the edge if the neighboring strands are directed anti-parallel.
   Otherwise replace the bonding site by a full-twist.
   The resulting diagram is a planar (multi-)knotoid diagram.
   Determine its type via knotoid invariants.

Twist Insertions



An application

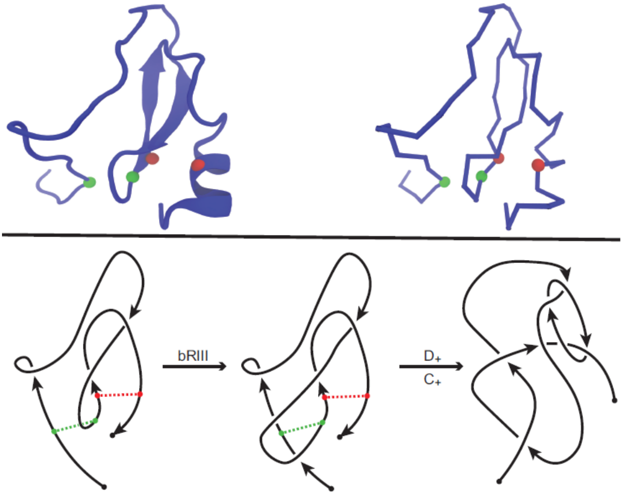


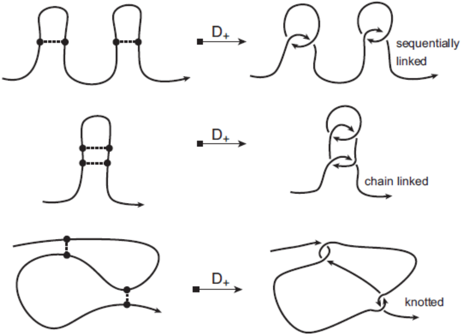
Figure: **A projection of the protein chain 2LFK**

Conclusion

With this model, we are able to detect three types of protein bonds

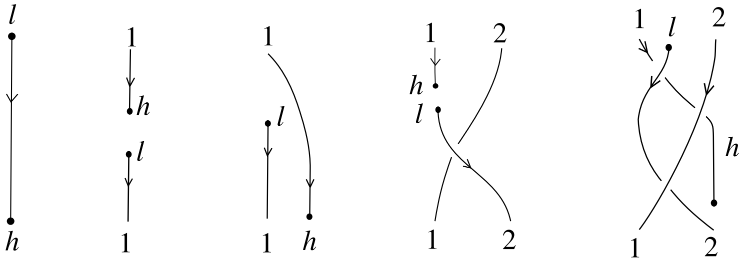
- sequential bonds
- nested bonds
- pseudoknot-like bonds

via knotoid invariants such as Turaev loop polynomial and the arrow polynomial.



The theory of braidoids

(Joint work with Lambropoulou)

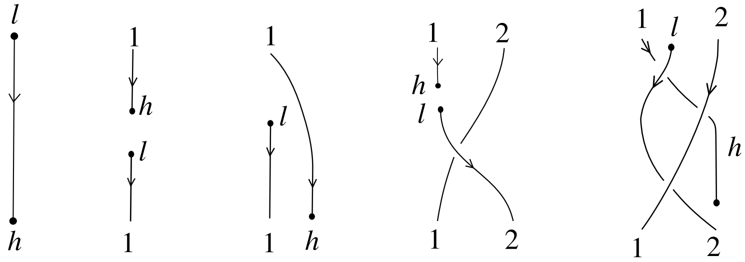


What is a braidoid diagram?

Definition

A *braidoid diagram* B is a system of a finite number of arcs embedded in $[0,1] \times [0,1] \subset \mathbb{R}^2$ that are called the *strands* of B.

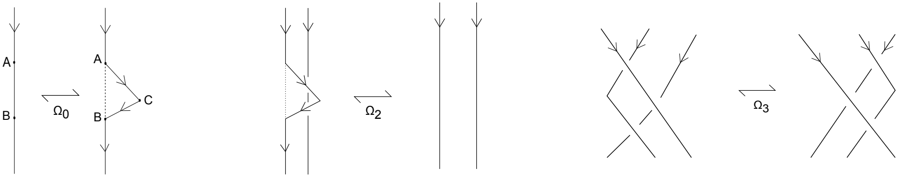
1. There are only finitely many intersection points among the strands, which are transversal double points endowed with over/under data, and are called *crossings*.
2. Each strand is naturally oriented downward, with no local maxima or minima, so that it intersects a strand horizontal line at most once.
3. A braidoid diagram has two types of strands, the classical strands and the free strands. A *free strand* has one or two ends that are not necessarily at $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. Such ends of free strands are called the *endpoints* of B.



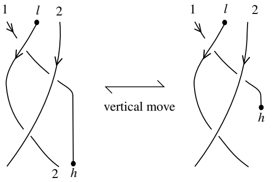
Examples of braidoid diagrams

Moves on braidoid diagrams

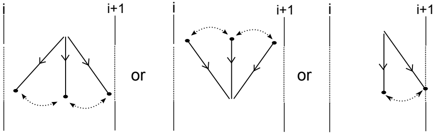
Δ-Moves:



Vertical Moves:



Swing Moves:



Braidoids

Definition

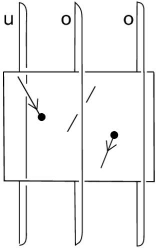
Two braidoid diagrams are said to be *isotopic* if one can be obtained from the other by a finite sequence of Ω-moves, vertical moves and swing moves. An isotopy class of braidoid diagrams is called a *braidoid*.

Definition

A *labeled braidoid diagram* is a braidoid diagram whose braidoid ends are labeled with o or u.
A *labeled braidoid* is an isotopy class of labeled braidoid diagrams up to the isotopy generated by the Ω-moves.

From a braidoid diagram to a knotoid diagram

- We define a closure operation on labeled braidoid diagrams by connecting each pair of corresponding ends accordingly to their labels and within a 'sufficiently close' distance:



The closure operation induces a well-defined map from the set of labeled braidoids to the set of planar multi-knotoids.

The analogue of the Alexander Theorem for braidoids

Theorem (The classical Alexander theorem)

Any classical knot/link diagram is isotopic to the closure of a classical braid diagram.

Theorem

Any multi-knotoid diagram in $\mathbb{R}^2$ is isotopic to the closure of a labeled braidoid diagram.

From a knotoid diagram to a braidoid diagram

We describe two braidoiding algorithms to prove our theorem.

The idea of the algorithms: To eliminate the up-arcs of a (multi)-knotoid diagram. We do this by the *braidoiding moves*:

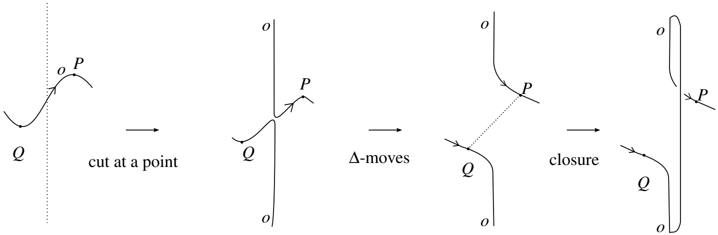
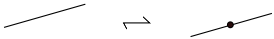


Figure: The germ of the braidoiding move and its closure

- Observe that the closure of each resulting labeled strand is isotopic to the initial up-arc.

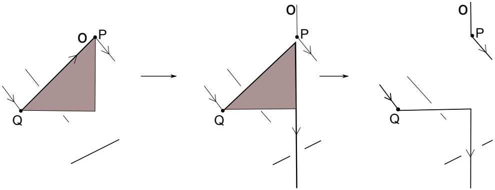
Preparatory notions for the braidoiding algorithms

- Subdivision:

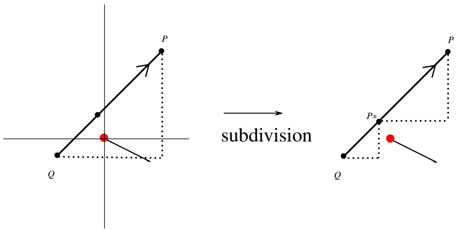


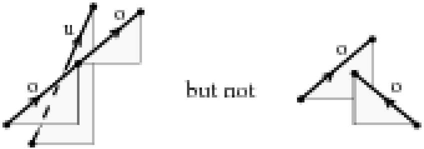
- Up-arcs and free up-arcs:

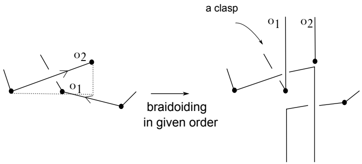
- Sliding triangles and the cut point:



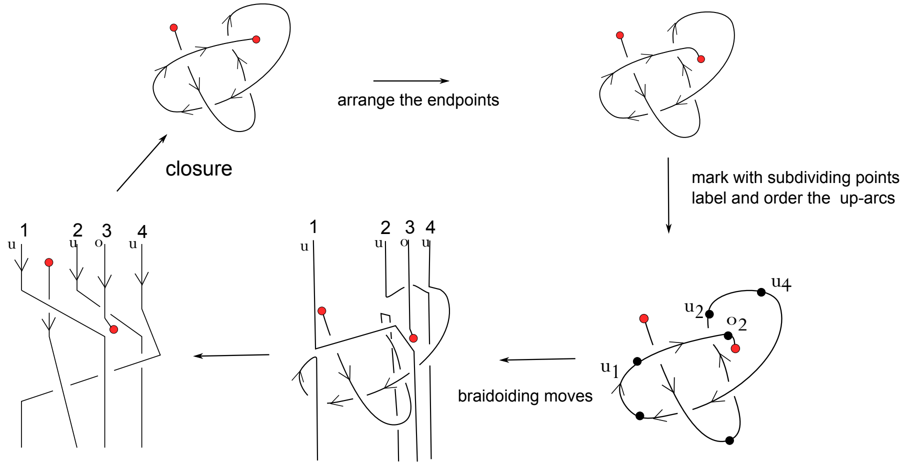
Triangle conditions



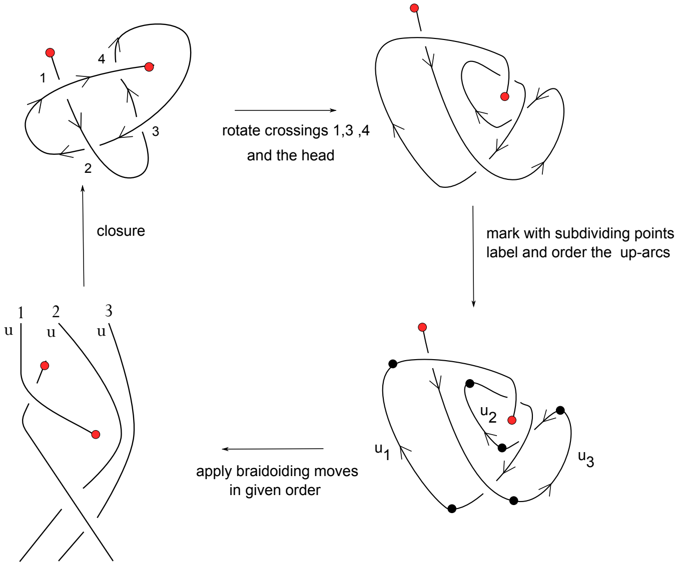




Braidoiding algorithm I:



Braidoiding algorithm II:



A corollary of the braidoiding algoithm II

Definition

A *u-labeled braidoid diagram* is a labeled braidoid diagram whose ends are labeled all with u.

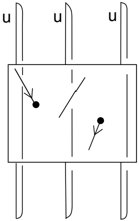
There is a bijection:

$$Label_u : \{\text{Braidoids}\} \rightarrow \{u\text{-labeled braidoids}\}$$

induced by assigning to a braidoid diagram a u-labeled braidoid diagram.

A sharpened version of the theorem

The uniform closure:



Theorem

Any multi-knotoid diagram $\mathbb{R}^2$ is isotopic to the uniform closure of a braidoid diagram.

Markov theorem for classical braids

Theorem (*Markov theorem*)

The closures of two braid diagrams b, b' in $\cup_{n=1}^{\infty} B_n$, represent isotopic links in $\mathbb{R}^3$ if and only if these braids are equivalent by the following operations.

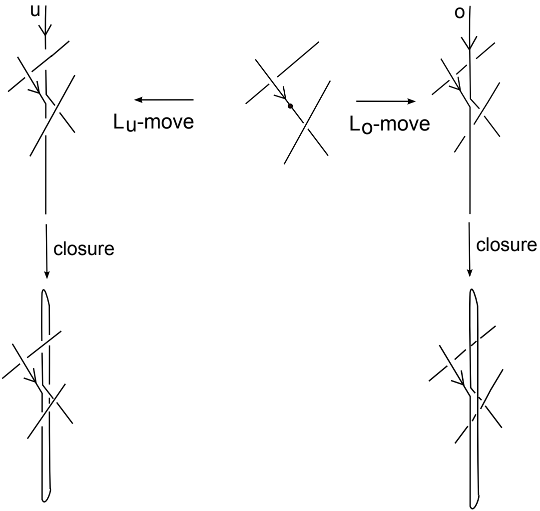
- *Conjugation: For $b, b' \in B_n$, $b' = gbg^{-1}$ for some $g \in B_n$.*
- *Stabilization: For $b \in B_n$, $b' \in B_{n+1}$, $b' = \sigma_n^{\pm} b$.*

Theorem (*One move Markov theorem*, Lambropoulou)

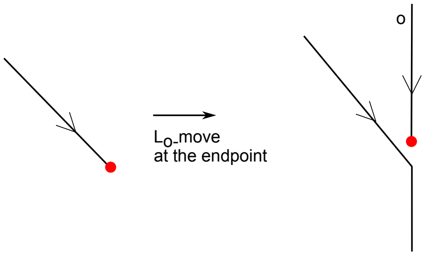
There is a bijection between the set of L-equivalence classes of braids and the set of isotopy classes of (oriented) link diagrams.

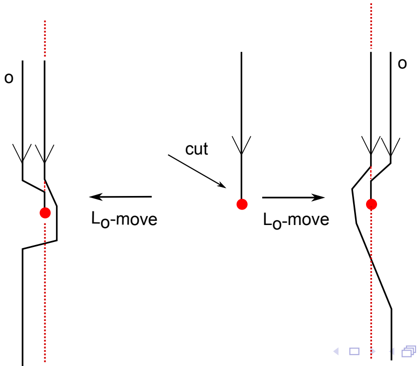
From knotoids to braidoids: Braidoid Equivalence

An *L-move* on a labeled braidoid diagram B is the following operation:



L-moves can be applied also on endpoints:





An analogue of the Markov theorem for braidoids

Definition

The L-moves together with labeled braidoid isotopy generate an equivalence relation on labeled braidoid diagrams that is called the *L-equivalence*.

Theorem

The closures of two labeled braidoid diagrams are isotopic (multi-)knotoids in $\mathbb{R}^2$ *if and only if the labeled braidoid diagrams are L-equivalent.*

A sketch for the proof

$\Rightarrow$: From our previous observation closure induces a well-defined map:
cl_L:$\{L$-eqv.classes of labeled braidoids$\} \rightarrow \{$Multi-knotoids$\}$.
$\Leftarrow$: The braidoiding algorithm I induces a well-defined map,

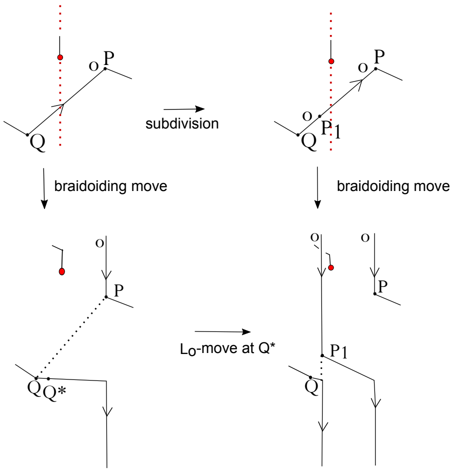
$$br: \{\text{Multi-knotoids in } \mathbb{R}^2\} \rightarrow \{L \text{ classes of labeled braidoids}\}.$$

For this we need to check:

- **Static Part**: Choices done for applying the algorithm such as subdivision, labeling of free up-arcs.
- **Moving Part**: The isotopy moves for knotoid diagrams including the moves displacing the endpoints.

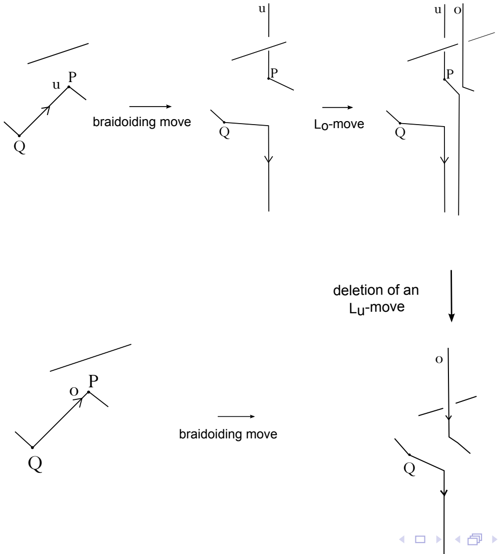
Lemma (Lambropoulou, Rourke)

Adding subdivision points to an up-arc results in L-equivalent braidoid diagrams.



Lemma (Lambropoulou, Rourke)

Changing the labeling of a free up-arc results in L-equivalent braidoid diagrams.

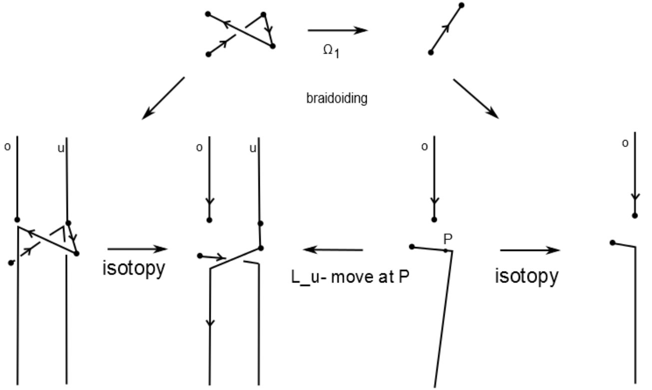


Corollary

Given any two subdivision S_1, S_2 of a knotoid diagram K with any admissible labeling, then the resulting braidoid diagrams are L-equivalent.

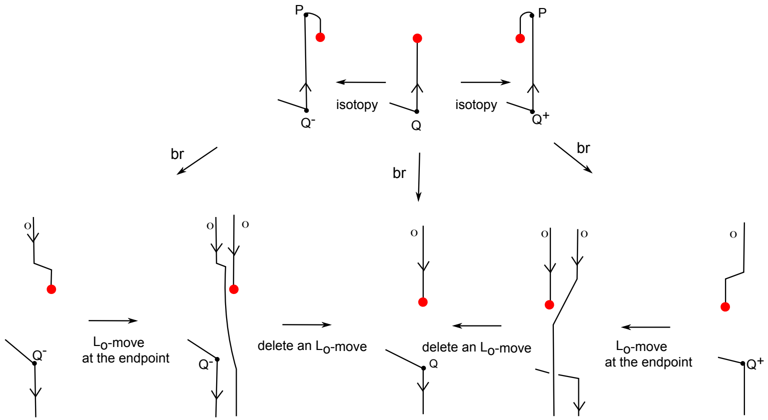
Lemma (Lambropoulou, Rourke)

Applying an Ω-move on a knotoid diagram results in L-equivalent braidoid diagram.



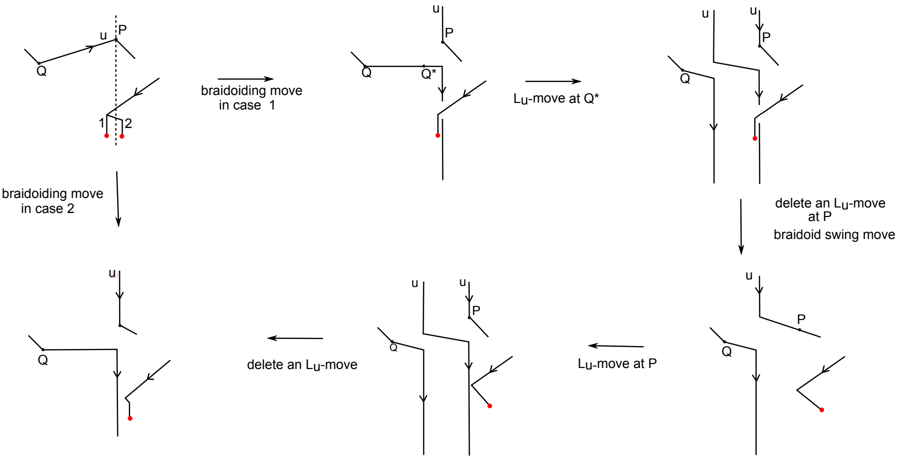
Lemma

Two ways to turn a vertical up-arc containing an endpoint result in L-equivalent braidoid diagrams.



Lemma

Planar isotopies displacing the endpoints result in L-equivalent braidoid diagrams.



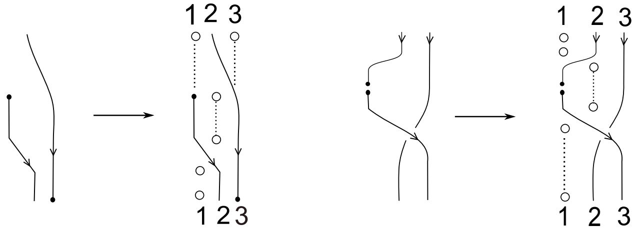
Corollary

The map br is well-defined, and also the maps br and cl_L are inverse maps of each other.

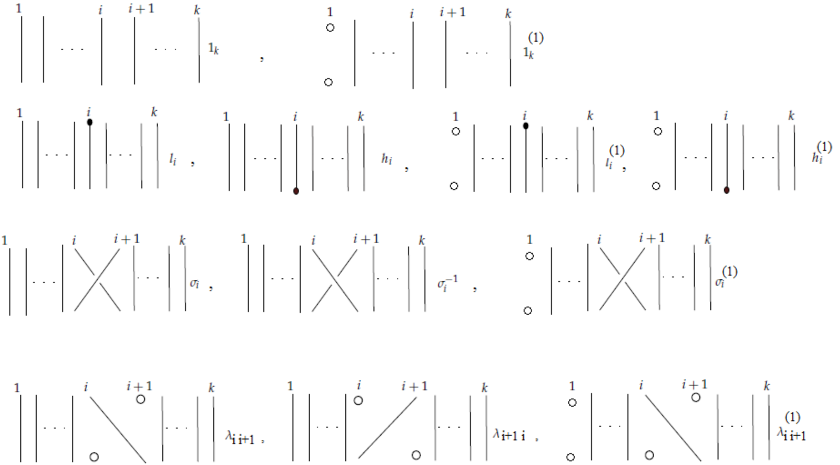
This completes the proof of our theorem.

Toward an algebraic structure

A braidoid diagram can be seen as a ‘composition’ of blocks of same size when it is filled with *implicit points*:

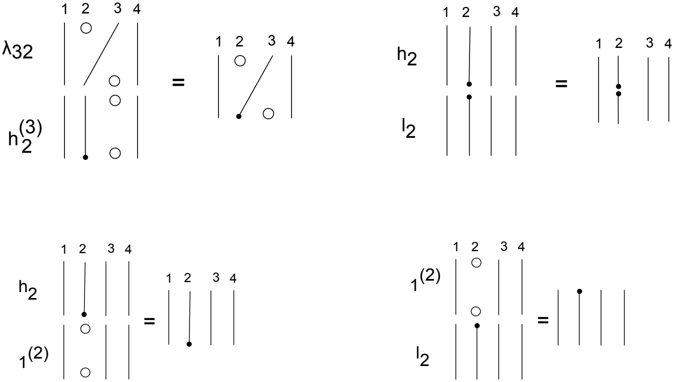


A braidoid diagram filled with implicit points is called a *combinatorial braidoid diagram*.
A combinatorial braidoid diagram can be decomposed into a finite number of blocks from the following set:



Composition rules

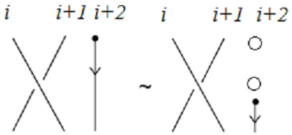
Conversely, a finite number of elementary blocks can be composed according to certain rules to form a combinatorial braidoid diagram:



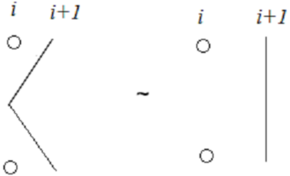
The composition of some 4-blocks

Relations on the composition

The composition is subject to some relations corresponding to the braidoid isotopy moves:



$\sigma_i^{\pm1} l_j = l_j (\sigma_i^{(j)})^{\pm1}$, for some j, $i+2 \leq j \leq k$ or $1 \leq j \leq i-1$



$\lambda_{ii+1}\lambda_{i+1i} = 1_k^{(i+1)}$

In this way we find a correspondence between the isotopy classes of braidoids to the set of combinatorial braidoids. This provides us an algebraic encoding of a braidoid.

Conclusions

In this PhD,

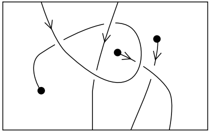
- We studied the theory of knotoids focusing on knotoids in S^2 and $\mathbb{R}^2$; we defined new invariants for them in analogy with the virtual knot invariants.
- We made a brief introduction to virtual knotoids.
- We gave a geometric interpretation for planar knotoids.
- We utilized our interpretation in topological analysis of protein chains.
- We initiated the theory of braidoids. We set the fundamental notions for braidoids and we studied them in relation to planar knotoids.

Most of the results have been presented in several conferences/meetings, and they have been published:

1. New invariants of knotoids
   (With L.H. Kauffman), European Journal of Combinatorics 65C (2017) pp. 186-229
2. Knotoids, braidoids and applications
   (With S.Lambropoulou), Symmetry 9(12):315, (2017)
3. On the height of knotoids
   (With L.H.Kauffman), to appear in Book at Springer Proceedings in Mathematics & Statistics (PROMS) titled: Algebraic Modeling of Topological and Computational Structures and Applications (2017)
4. Topological models for knotted bonded open protein chains using the concept of knotoids and bonded knotoids
   (With D.Goundaralis, S.Lambropoulou, J.Dorier, A.Stasiak and L.H.Kauffman) Polymers 9(9), 2017)

Further Questions/Directions

1. The next step to the study of braidoids is to understand the algebraic structure admitted by braidoids.
2. The elementary blocks of a braidoid diagram are in resemblance with Rook diagrams.
   Is there an association of the set of braidoids with a diagram algebra such as Rook or Motzkin algebras ?.
3. We can define a (oriented) tangloid diagram by extending the notion of a tangle with presence of ‘free’ strings:



An oriented tangloid

We will make a further study on tangloids .

Thank you for your attention!