Parity in Knotoids

Neslihan Gügümcü project in progress w/ Louis H.Kauffman*

National Technical University of Athens,
Department of Mathematics

*University of Illinois at Chicago,
Department of Mathematics

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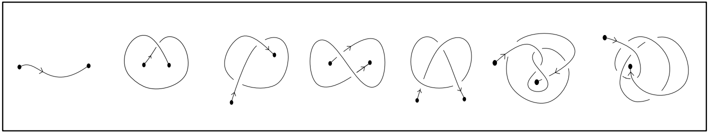
What is a knotoid?

Definition [Turaev, 2012]: A *knotoid diagram* K in S^2 or $\mathbb{R}^2$ is simply a knot diagram with underlying curve is the unit interval $[0,1]$ rather than a circle.

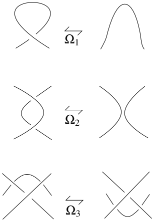
- The endpoints of K are called the *tail* and the *head*, respectively.
- The tail and the head can be located in any region of the diagram.
- K is oriented from its tail to its head.



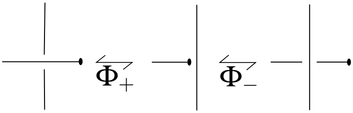
Knotoid diagrams

What is a knotoid?

Definition: A *knotoid* is an equivalence class of the knotoid diagrams up to the equivalence relation induced by $\Omega_{i=1,2,3}$- moves plus the isotopy of S^2 (or $\mathbb{R}^2$, respectively).



Equivalence moves



Forbidden knotoid moves

What is an invariant of a knotoid?

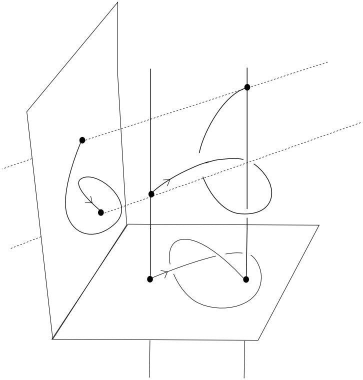
Definition: Let M denote a set of mathematical objects. An *invariant of knotoids* is a mapping

$$I : \{\text{Knotoid diagrams}\} \to M,$$

assigning the same object to equivalent knotoid diagrams.

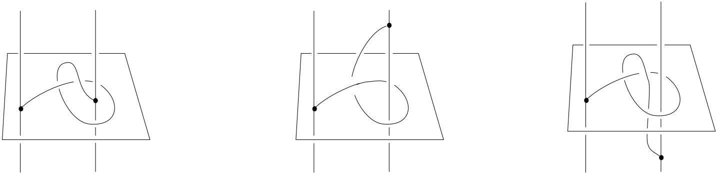
A geometric interpretation

A knotoid diagram in $\mathbb{R}^2$ can also be seen as a generic projection of an open-ended smooth oriented curves embedded in $\mathbb{R}^3$ along the two lines passing through the endpoints of the curve.



A geometric interpretation

Any knotoid determines an open-ended oriented curve in $\mathbb{R}^3$: Keep the endpoints attached on the two lines passing through the endpoints and perpendicular to the plane of a knotoid diagram and push the crossings up or down in the vertical direction accordingly to their over/under-data.



A geometric interpretation of knotoids in $\mathbb{R}^2$

Theorem (G. - Kauffman)

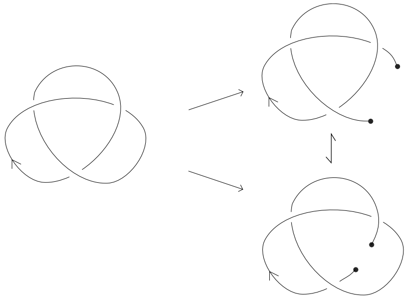
Two open-ended oriented curves embedded in $\mathbb{R}^3$ that are both generic to a given plane, are line isotopic (with respect to the lines determined by the endpoints of the curves and the plane) if and only if the projections of the curves to that plane are equivalent knotoid diagrams in the plane.

Knotoids as an extension of classical knots

- The theory of knotoids in S^2 is an extension of classical knot theory. The map α,

$$\alpha\colon\ \text{Classical knots} \to \text{Knotoids in } S^2,$$

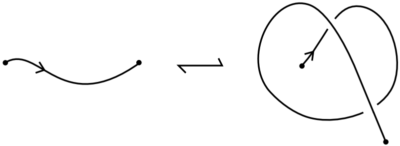
is induced by deleting an open arc which does not contain any crossings from an oriented classical knot diagram. The map α is injective.



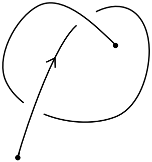
Knotoids as an extension of classical knots

Definition: A knotoid in S^2 that is in the image of α, is called a *knot-type* knotoid. A knotoid that is not in the image of α is called a *proper* knotoid.

$$\{\text{Knotoids in } S^2\} = \{\text{Knot-type knotoids}\} \cup \{\text{Proper Knotoids}\}$$



A knot-type knotoid



A proper knotoid

Evidences of parity in knotoids

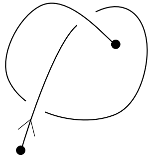
Definition: A *parity* for the set of knotoids is a rule associating 0 or 1 with every crossing of any knotoid diagram such that:

1. the parity of the crossings outside any $\Omega_{i=1,2,3}$- move region remain the same.
2. The sum of the parities of the three crossings in a Ω_3-move is zero modulo 2.

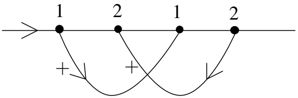
Gauss codes/diagrams for knotoids

Definition: The *Gauss code* of a knotoid diagram K is a linear code that consists of a sequence of labels each of which is assigned to the crossings encountered during a trip along K from its tail to the head. The *Gauss* diagram of K is a diagram corresponding to the Gauss code of K with:

1. a line segment oriented from left to right with $2n$ labeled points placed on each corresponding to the crossings of K
2. oriented and signed chords connecting the same labels of crossings.



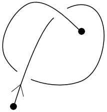
$O1+U2+U1+O1+$



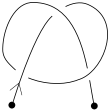
The Gaussian parity for knotoids

Definition: Let K be a knotoid diagram. A crossing of K is called *even* if there is an even number of labels between two appearances of the label of the crossing in the Gauss code of K.
A crossing of K is called *odd* if there is an odd number of labels between two appearances of the label of the crossing in the Gauss code of K.

Example:



$O1+U2+U1+O2+$



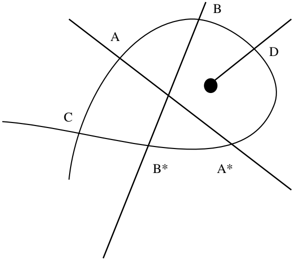
$O1+U2+03+U1+02+U3+$

Important fact: Any crossing of a knot-type knotoid diagram is even.

The Gaussian parity

Lemma (G.–Kauffman)

A knotoid diagram consists in only even crossings if and only if it is a knot-type knotoid diagram.



A loop at a crossing in a proper knotoid diagram

An invariant of parity: Odd writhe

Definition:

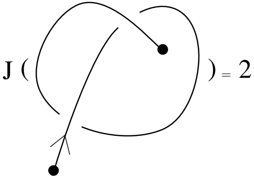
$$\textit{Odd Writhe} \text{ of } K = J(K) = \sum_{c \in \mathrm{Odd}(K)} \mathrm{sign}(c),$$

where K is a knotoid diagram and Odd(K) is the set of odd crossings in K.

Theorem (G.–Kauffman)

Odd writhe is a knotoid invariant.

Corollary: Odd writhe of a knot-type knotoid is zero.



A proper knotoid with non-zero odd writhe

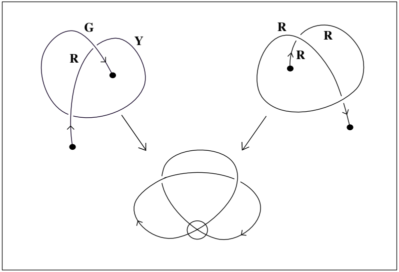
From knotoids to virtual knots

Every knotoid diagram in S^2 represents a virtual knot:
There is well-defined map induced map called the *virtual closure map* induced by connecting the endpoints in the virtual fashion:

$$\bar{v}\colon \text{Knotoids in } S^2 \to \text{Virtual knots of genus} \leq 1$$

The virtual closure map

- The virtual closure map is not injective.



A pair of nonequivalent knotoids with the same virtual closure

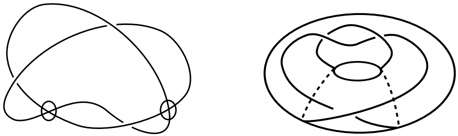
The virtual closure map

Proposition (G.–Kauffman)

The virtual closure map is not surjective.

A sketch of the proof

We claim that the following virtual knot which is of genus 1, is not in the image of the virtual closure map:

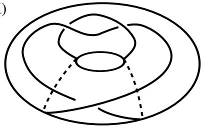


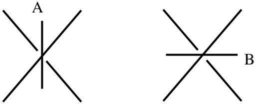
This can be shown by examining the surface-state curves of the diagram in torus.

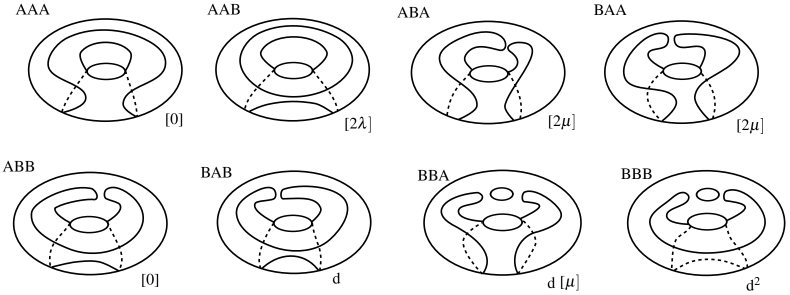
Sketch of the proof

<u>Observation 1</u>: Non-trivial state curves of the following knot diagram are of the form $2[\lambda]$ and $2[\mu]$.

(T,K)

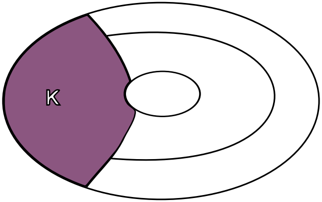






A sketch for the proof

Main observation:
Let be given the standard torus representation of the virtual closure of a knotoid diagram K.
By construction, each of the surface states of this representation has curves of the form $[\lambda]+a[\mu]$, where λ and μ are the longitude and the meridian of T^2, and $a \in \mathbb{Z}$.



A knot diagram in T^2 that is a virtual closure of a knotoid diagram K

A sketch of the proof

Fact:

If a virtual knot diagram k represents a genus 1 knot that is in the image of the virtual closure map, then any torus representation of k has a state curve in the form $h_*([\lambda]+a[\mu])$, for some $h \in Aut(T^2)$ and h_* is the induced isomorphism on $H_1(T^2, \mathbb{Z})$.

End of the proof

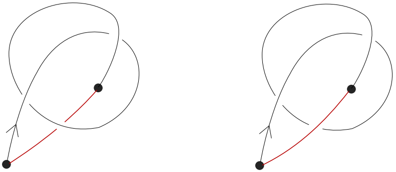
No orientation preserving homeomorphism of torus can take the curve $[\lambda]+a[\mu]$ to either of the loops of the form $2[\lambda]$ or $2[\mu]$.

Conclusion: The virtual knot in question is not in the image of the virtual closure map. Therefore, the virtual closure map is not surjective.

From knotoids to classical knots

There is a well-defined map surjective induced by connecting the endpoints of a knotoid diagram by an underpassing arc,

$$\omega_{-}\colon \text{Knotoids in } S^2 \to \text{Classical knots}$$



Two types of closures resulting in different classical knots

The height of a knotoid

Definition:

The *height* of a knotoid diagram is the minimum number of crossings that a connection arc creates during the underpass closure.

The *height of a knotoid K* is defined as the minimum of the heights, taken over all equivalent knotoid diagrams to K.

Theorem (Turaev)

The height is a knotoid invariant.

A knotoid has zero height if and only if it is a knot-type knotoid.

$$\equiv$$

A knotoid has non-zero height if and only if it is a proper knotoid.

Some estimations of height

Theorem (G. – Kauffman)

Let K be a knotoid diagram and m be the maximum degree of the affine index polynomial of K. Then the height of K, $h(K) \geq m$.

Theorem (G. – Kauffman)

The height of a knotoid K in S^2 is greater than or equal to the Λ-degree of its arrow polynomial.

A conjecture on the height

Conjecture (Turaev)

Minimal crossing diagrams of knot-type knotoids have zero height.

A recipe to prove the conjecture

Main Ingredient:

Theorem (Manturov)

Let κ be an isotopy class of a classical knot. Then the minimal number of classical crossings for virtual diagrams of κ is realized on classical diagrams (genus 0- diagrams) up to detour moves.

A recipe to prove the conjecture

Main Ingredient:

Theorem (Manturov)

Let κ be an isotopy class of a classical knot. Then the minimal number of classical crossings for virtual diagrams of κ is realized on classical diagrams (genus 0- diagrams) up to detour moves.

Sketch of the proof:

1. Let k be a knot-type knotoid then $\overline{v}(k)$ is a classical knot.
2. Assume there exists a minimal crossing knotoid diagram K with nonzero height. This forces the virtual knot diagram $\overline{v}(K)$ to be a minimal diagram for $\overline{v}(k)$.
3. The underlying genus of $\overline{v}(K)$ is 1.
4. This contradicts with the above theorem.

Corollary

Crossing number of a knot-type knotoid is equal to the crossing number of the knot that is the underpass closure of the knotoid.

More conjectures on the height

The virtual crossing number of the virtual closure of a knotoid is less than or equal to the height of the knotoid.

Conjecture (G.–Kauffman)

Let κ be a virtual knot that is the virtual closure of a knotoid K in S^2. The virtual crossing number of κ is equal to the height of K.

Conjecture (G.–Kauffman)

There is no proper knotoid whose virtual closure is a classical knot.

Thank you for your attention!