Invariants of Knotoids

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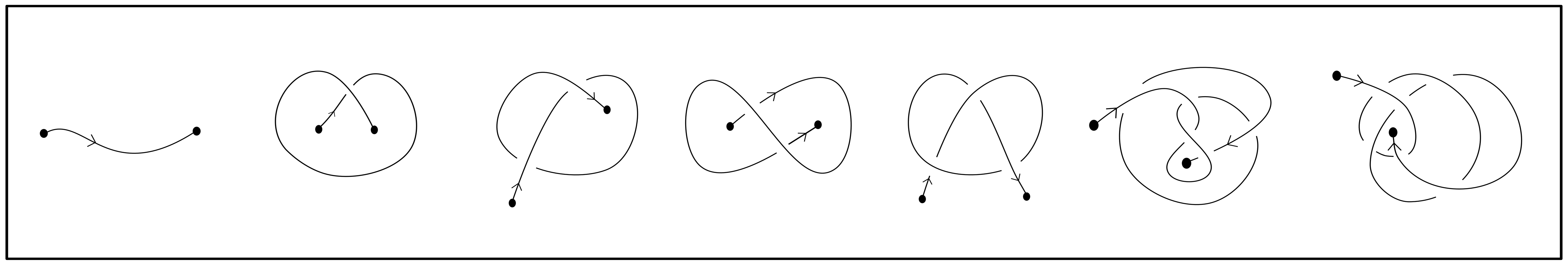


Background

Knotoids are defined by V. Turaev in 2010 [3].

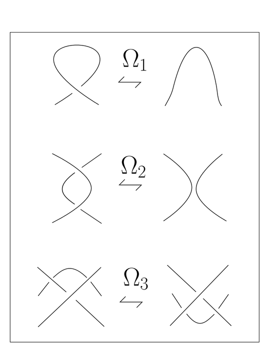
- A new diagrammatic approach to knot theory: Every knotoid diagram represents a classical knot. Connect the endpoints by a simple arc which goes under every strand it meets, this operation is called the *underpass closure*.
- An extension of classical knot theory: Cut out an open segment of an oriented knot diagram which is disjoint from the crossings, to obtain a knotoid diagram with endpoints lying in the same local region. This defines an injective map from the set of classical knots to the set of classical knotoids.

Definition

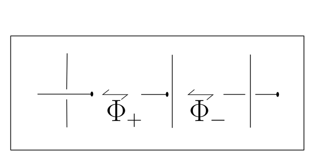
A knotoid diagram K in S^2 is a generic immersion $K : [0,1] \to S^2$ with

1. finitely many transversal double points, named as *crossings* of K,
2. two endpoints as the images of 0 and 1, named as the *tail* and the *head*, respectively.

The endpoints are distinct from each other and from the crossings of K. The orientation is always from the tail to the head.



Together with isotopy of S^2, $\Omega_{i=1,2,3}$- moves generate an equivalence relation on knotoid diagrams



It is forbidden to pull an endpoint over/under a transversal strand!

A *classical knotoid* is an equivalence class of all equivalent knotoid diagrams.

The *height* (or the complexity) of a classical knotoid diagram is the minimum number of crossings that a connection arc creates during the underpass closure. The *height of a classical knotoid* K is defined as the minimum of the heights, taken over all equivalent classical knotoid diagrams to K. The height is a classical knotoid invariant [3].

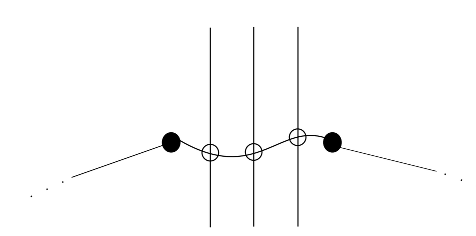
The Virtual Closure

Every classical knotoid diagram represents a virtual knot:

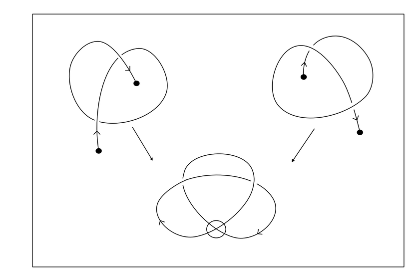
The *virtual closure* map

$\overline{v}$: Classical Knotoids $\to$ Virtual Knots of genus ≤ 1

is defined by connecting the endpoints of a classical knotoid diagram in the virtual way.



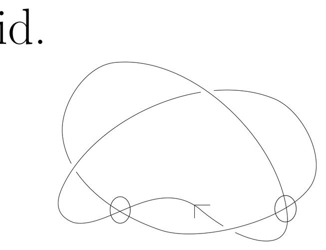
The virtual closure of a knotoid diagram



The virtual closure map is not injective!

The bracket polynomial detects the non-equivalency of the knotoids in the figure above. The Jones polynomial extends to knotoids via the normalized bracket polynomial of knotoids [3].

Conjecture 1:The Jones polynomial,$J(K)$, detects the trivial knotoid.



This is a virtual knot with trivial Jones polynomial [2]

A fact: $J(K) = J(\overline{v}(K))$, for any classical knotoid K.

Conjecture 2: The virtual knot in the figure is not the image of $\overline{v}$.

Motivation: The Great Connection

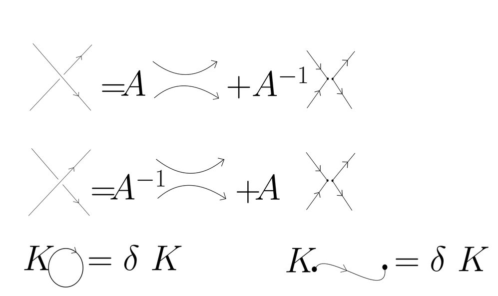
The virtual closure map, being a well-defined map, is a key to apply virtual knot invariants to knotoids. In fact, if $\mathrm{Inv}(K)$ denotes a virtual knot invariant then we can define a knotoid invariant $\mathrm{Inv}(K)$ for any classical knotoid K by the following formula, $\mathrm{Inv}(K) = \mathrm{Inv}(\overline{v}(K))$.

Invariants of knotoids that are also virtual knot invariants

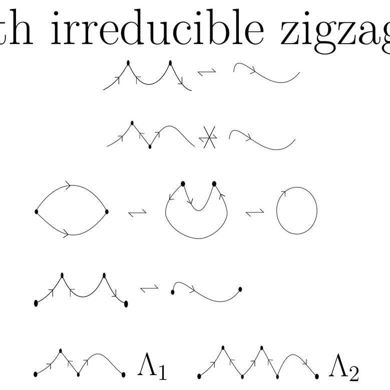
The *arrow polynomial* of a classical knotoid diagram K, $A[K]$ depends on the oriented state expansion.

$$A[K] = \Sigma_S < K|S > (-A^2 - A^{-2})^{\|S\|-1}\Lambda_i$$

where the sum runs over the oriented bracket states, $< K|S >$ is the usual vertex weights of the bracket polynomial, $\|S\|$ is the number of components of the state S and Λ_i is the variable associated to the long segment component of S with irreducible zigzags.



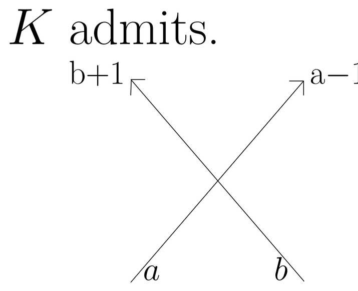
Oriented state expansion, $\delta = (-A^2 - A^{-2})$



Reduction rules: The circular components are free of zig-zags, long components can have irreducible zig-zags!

The Λ-*degree* of the index polynomial is the max. of $i \in \mathbb{Z}$ assigned to Λ_i appearing in the polynomial.

The *affine index polynomial* of a knotoid K, $P_K(t)$ depends on an integer labeling that the underlying flat diagram of K admits.



Integer labeling at a flat crossing

Start labeling the arc next to the tail with 0. Each arc of the flat diagram of K is labeled according to the rule. Then at each crossing c of K we have two number outcomes:

$$w_+(c) = a - (b+1)$$
$$w_-(c) = b - (a-1).$$

The *weight* of c is:

$$w_K(c) = \begin{cases} w_+(c), & \text{if the sign of } c \text{ is a positive,} \\ w_-(c), & \text{if the sign of } c \text{ is a negative} \end{cases}$$

$$P_K(t) = \Sigma_c \operatorname{sign}(c)(t^{w_K(c)} - 1)$$

The height of a knotoid can be estimated both by the affine index and the arrow polynomial.

Result 1

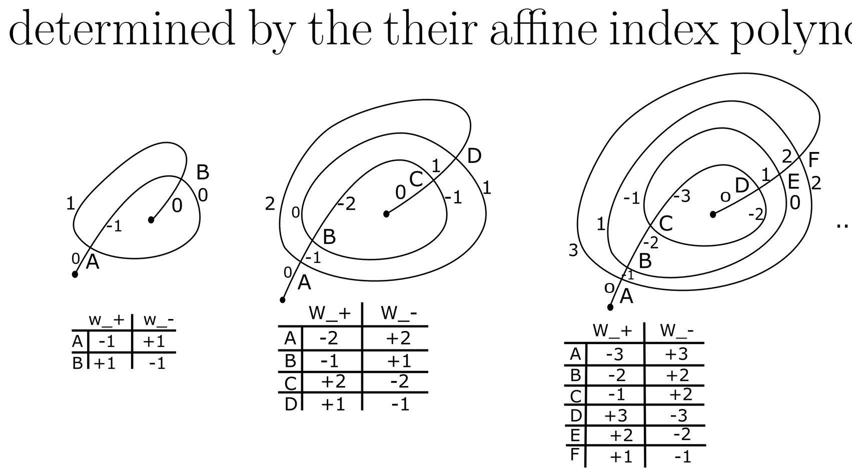
Let K be a classical knotoid and $|m|$ be the maximal weight of the crossings of K. If the maximum degree of the affine index polynomial of K is $|m|$ then the height of K, $h(K) \geq |m|$.

Result 2

The height of a classical knotoid K is greater than or equal to the Λ-degree of its arrow polynomial.

Which lower bound is better?

There is a pair of crossings c_1, c_2 of the n-fold flat spiral knotoid diagram such that $(w_+(c_1), w_+(c_2)) = (n, -n)$. Then the heights of the knotoids overlying the n-fold flat diagrams with all crossings positive, are determined by the their affine index polynomials.



1, 2 and 3-fold flat spirals

If the crossings of the 3-fold spiral, A, B, C are positive and D, E, F are negative then the index polynomial gives trivial lower bound for the height. Fortunately the Λ-degree of the arrow polynomial is 3. Therefore this knotoid has height 3. We have many other cases in which the arrow polynomial gives a better estimation for the height.

References

[1] N.Gügümcü, L.H.Kauffman,*New Invariants of Knotoids*, `http://arxiv.org/abs/1602.03579`

[2] L.H.Kauffman, *Introduction to Virtual Knot Theory*, JKTR, **21**, (2012), no.13, 37 pp.

[3] V.Turaev, *Knotoids*, Osaka J.of Math., **49**, (2012), no.1, 195-223

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