How to estimate the height of a knotoid?

Knots in Hellas'16

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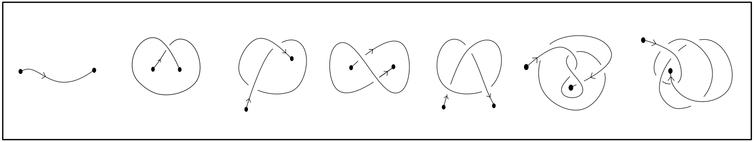
What is a knotoid?



Definition

A *knotoid diagram* in S^2 is a generic immersion

$$K : [0,1] \to S^2$$

1. endowed with under/over-data at each of its singularities that we call *crossings*.
2. the images of 0 and 1 are regarded as the endpoints and they are distinct from the crossings and each other.

What is a knotoid?

The three Reidemeister moves which do not interfere with the endpoints generate an equivalence relation on knotoid diagrams.

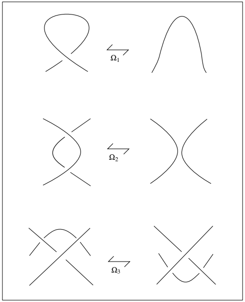


Figure: **The equivalence moves**

A *knotoid* is an equivalence class of the knotoid diagrams up to this equivalence relation.

Note that allowing the following two moves trivialize all diagrams.

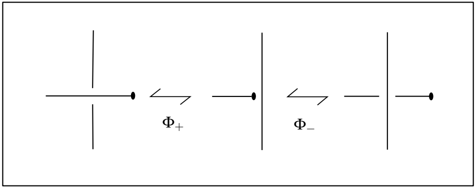


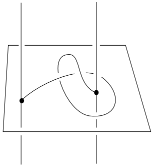
Figure: **Forbidden moves!**

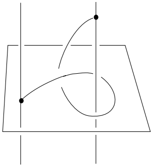
A Geometric interpretation

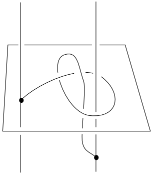
Definition

Two smooth open oriented curves embedded in $\mathbb{R}^3$ with the endpoints that are attached to two special lines, are said to be *line isotopic* if there is a smooth ambient isotopy of the pair $(\mathbb{R}^3 \setminus \{t \times \mathbb{R}, h \times \mathbb{R}\}, t \times \mathbb{R} \cup h \times \mathbb{R})$ such that

- taking one curve to the other curve in the complement of the lines,
- taking endpoints to endpoints,
- taking lines to lines; $t \times \mathbb{R}$ to $t \times \mathbb{R}$ and $h \times \mathbb{R}$ to $h \times \mathbb{R}$.







- There is a bijection between the set of knotoids and the line isotopy clases of smooth open oriented curves in $\mathbb{R}^3$.

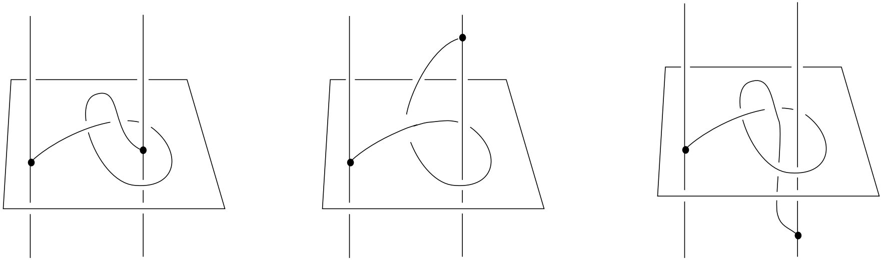


Figure: **Curves in $\mathbb{R}^3$ obtained by a knotoid**

Knotoids vs Classical Knots

- Knotoids give a new diagrammatic approach to knot theory: Every knotoid diagram represents knots in $\mathbb{R}^3$ via the *over*- or *under-closure*.

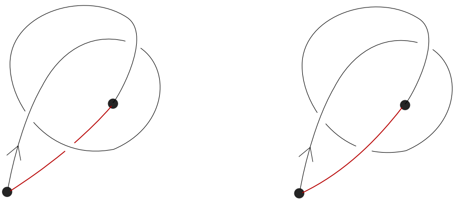


Figure: **Two types of closures**

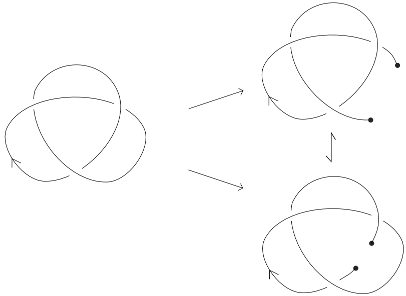
Any knot in $\mathbb{R}^3$ can be represented by a knotoid diagram:
Cut out an underpassing arc from a given oriented diagram of the knot to obtain a knotoid diagram of the knot.

Classical Knots vs Knotoids

- Knotoids form an extension of the classical knot theory: There is an injective map

$$\alpha: \text{ The classical knots in } S^2 \to \text{Knotoids in } S^2$$

given by deleting an open arc which does not contain any crossings from a given oriented knot diagram.



The set of knotoids $\mathscr{K}(S^2)$ =Knot-type knotoids$\cup$Proper Knotoids

The height of a knotoid

Definition

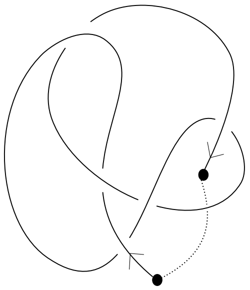
The *height* of a knotoid diagram is the minimum number of crossings that a shortcut creates during the underpass closure. The *height of a knotoid K* is defined as the minimum of the heights, taken over all equivalent knotoid diagrams to K.

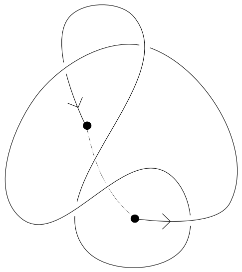
Theorem

The height is a knotoid invariant.

A knotoid has zero height if and only if it is a knot-type knotoid.
$\equiv$ A knotoid has non-zero height if and only if it is a proper knotoid.

How can we compute the height?



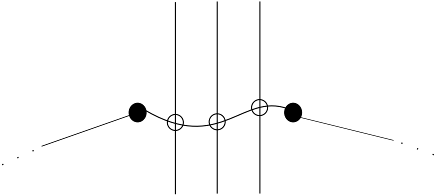


Question

Apparently, the first diagram has height 1 and the second one has height 2. But are there some equivalent diagrams to the knotoids above with less height?

The virtual closure

Every knotoid diagram in S^2 represents a virtual knot as well: The endpoints of a knotoid diagram are connected by an embedded arc in S^2 which is chosen to create a virtual crossing whenever it meets with a strand of the diagram.

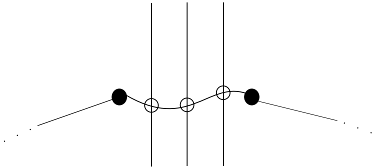


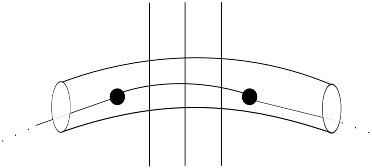
The virtual closure

Definition

This connection operation is called *virtual closure* of knotoids and defines a well-defined map,

$$\overline{v}\colon \text{Knotoids in } S^2 \to \text{Virtual Knots of genus} \leq 1$$





The virtual closure

- The virtual closure map is not injective:

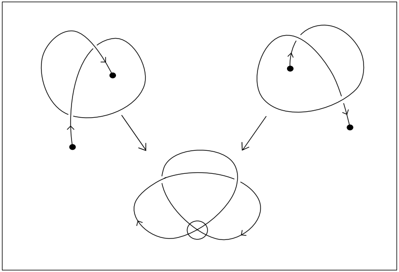


Figure: These knotoids can be distinguished by their bracket polynomials

- For any knotoid K, $\langle K \rangle = \langle \overline{v}(K) \rangle$.

Question

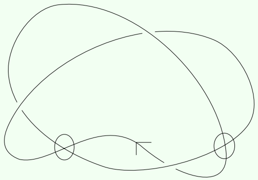
Is the virtual closure map surjective?

Conjecture

The Jones polynomial detects the trivial knotoid.

A comment

A genus 1 virtual knot with trivial Jones polynomial is depicted below.
If the above conjecture holds then this knot can not be in the image of $\overline{v}$. Thus $\overline{v}$ can not surjective.



Our Motivation for defining new invariants for knotoids

Since $\overline{v}$ is a well-defined map, for $Inv(K)$ any invariant of virtual knots we can define a knotoid invariant $Inv(K)$ for any knotoid such that

$$\mathrm{Inv}(K) = \mathrm{Inv}(\overline{v}(K))$$

An affine index polynomial of knotoids

K is a knotoid diagram, we label each *arc* of the corresponding flat diagram to K by the following rule:

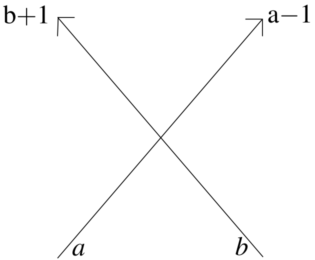


Figure: **Integer labeling at a flat crossing**

An affine index polynomial of knotoids

Definition

For c is a crossing of K, two number outcomes are derived, $w_+(c)$ and $w_-(c)$ which are called *positive* and *negative* weights of c, and defined as,

$$w_+(c) = a - (b+1)$$
$$w_-(c) = b - (a-1).$$

where a and b are the labels for the left and the right incoming arcs at the corresponding flat crossing to c, respectively.
The *weight* of c is defined as

$$w_K(c) = \begin{cases} w_+(c) & \text{if the sign of } c \text{ is positive,} \\ w_-(c) & \text{if the sign of } c \text{ is negative} \end{cases}$$

An affine index polynomial of knotoids

Definition

The *affine index polynomial* of a knotoid K is defined by the equation,

$$P_K(t) = \sum_c \operatorname{sign}(c)(t^{w_K(c)} - 1)$$

where the sum is taken over all crossings of a diagram of K.

Theorem

The affine index polynomial is a knotoid invariant.

A comparison of the affine index polynomials: Knots vs Knotoids

Theorem (Ka.)

The affine index polynomial is trivial for all classical knots.

A conclusion

Knot-type knotoids have trivial affine index polynomial.
If a given knotoid diagram has nonzero affine index polynomial, then the knotoid diagram represents a proper knotoid.

A comparison of the affine index polynomials

- The affine index polynomial can be used to distinguish a virtual knot from its inverse.

Theorem (Ka.)

For the affine index polynomial of a virtual knot k, we have

$$P_K(t) = P_{\overline{K}}(t^{-1}),$$

where K is an oriented diagram of k and $\overline{K}$ is the inverse of K.

This property of the affine index polynomial does not hold for knotoids.

A comparison of the affine polynomials

Theorem (Gu., Ka.)

The affine index polynomial of a knotoid is symmetric with respect to $t \leftrightarrow t^{-1}$, that is, $P_K(t) = P_{\overline{K}}(t)$.

- $P_K(t) = P_{\overline{v}(K)}(t)$ for any knotoid K.

A conclusion

If the affine index polynomial of a virtual knot is not symmetric with respect to $t \leftrightarrow t^{-1}$ then it is not in the image of the virtual closure map $\overline{v}$.

But a genus 1 virtual knot with a non-symmetric affine index polynomial is still wanted!

The affine index polynomial and the height of knotoids

Theorem (Gu.,Ka.)

Let K be a knotoid diagram and $|m|$ be the maximal weight of the crossings of K. If the maximum degree of the affine index polynomial of K is $|m|$ then the height of K, $h(K) \geq |m|$.

Sketch of the proof of the theorem

- Every crossing of a knotoid diagram forms a loop. Let C be the crossing with maximal weight and $l(C)$ be the associated loop to C.
  Algebraic int. number of $l(C)$ with the rest of the diagram is equal to either $w_+(C)$ or $w_-(C)$.

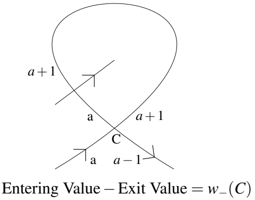


Figure: **The weights with respect to the orientation of the loop at C**

Sketch of the proof

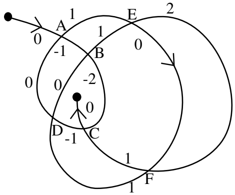
- Apply the *Seifert smoothing* to all crossings which are met twice while traveling through the loop $l(C)$. The Seifert smoothing of the crossing C results in oriented Seifert circles and an oriented long segment containing the endpoints.

Figure: **Seifert smoothing of a crossing**

Sketch of the proof

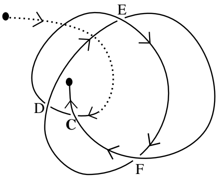
- Observe that
$$|I_K| = |w(C)| = |m| \leq \# \text{ of Seifert circles}$$
The height of the diagram by the Jordan curve theorem,
$$h(K) \geq \# \text{ of Seifert circles}$$
Thus we have,
$$h(K) \geq |m|,$$
where $|m|$ is the maximal weight of the crossings of K.
- The rest holds from the invariance of the polynomial.

A visualization of the proof



	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	1	1
F	1	-1

$$P_K(t) = t^2 + t + t^{-1} + t^{-2} - 6$$



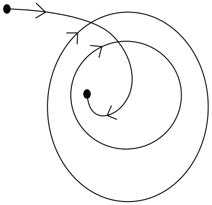


Figure: **The loop of *C* and the Seifert circles&the long segment**

The arrow polynomial

The construction of the arrow polynomial of knotoids is based on the *oriented state expansion* of the bracket polynomial of knotoids.

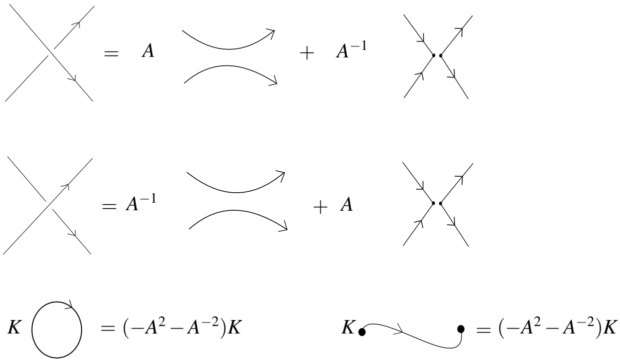
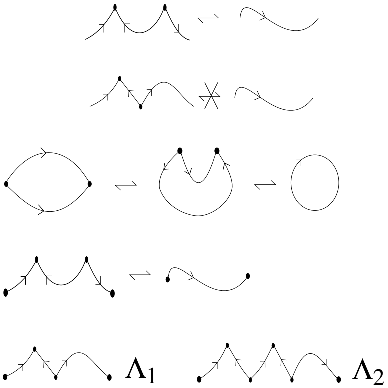


Figure: **Oriented state expansion**

Reduction rules for the arrow polynomial

To reduce the number of cusps in a state component we have the following rules:



Definition

We define the *arrow polynomial* of a knotoid diagram K as,

$$\mathscr{A}[K] = \sum_S < K|S > (-A^2 - A^{-2})^{\|S\|-1} \Lambda_i$$

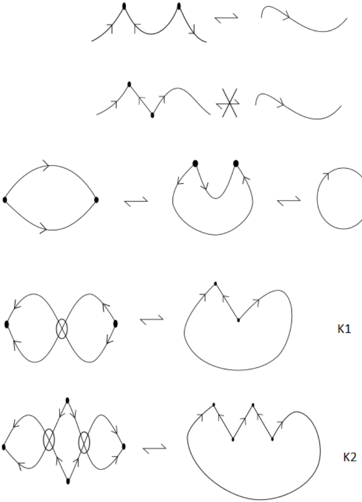
where the sum runs over the oriented bracket states, $< K|S >$ is the usual vertex weights of the bracket polynomial, $\|S\|$ is the number of components of the state S and Λ_i is the variable associated to the long segment component of S with irreducible zig-zags.

Theorem

The normalization of the arrow polynomial (with the writhe) is a knotoid invariant.

Some notes from VKT

The normalized arrow polynomial is defined originally as a virtual knot invariant by L.Kauffman and H.Dye, based on the oriented state expansion.



Definition

The *K-degree* of a summand of the arrow polynomial of a knot which is of the form, $A^m({K_{i_1}}^{j_1}{K_{i_2}}^{j_2}\ldots{K_{i_n}}^{j_n})$ is equal to

$$i_1 \times j_1 + \ldots i_n \times j_n.$$

The *K-degree* of the arrow polynomial of a virtual knot is the maximum K- degree of the polynomial.

Proposition (Ka)

The arrow polynomials of classical knots are free of K-variables.

Definition

The *virtual crossing number* of a virtual knot/link is the minimum number of virtual crossings over all representative diagrams.

Theorem (Dy.,Ka.)

The virtual crossing number of a virtual knot/link is greater than or equal to the K-degree of the arrow polynomial of that virtual knot/link.

The arrow polynomial and the height

Definition

The Λ-*degree* of a summand of the arrow polynomial of a knotoid is equal to the half of the number of irreducible cusps in the corresponding long state component.
The Λ-*degree* of the arrow polynomial of a knotoid is the maximum Λ- degree of the polynomial.

A note

Circular components of states of a knotoid diagram K are free of irreducible zig-zags.
The Λ_i-degrees of the $\mathscr{A}(K)$ turn into the K_i-degrees of $\mathscr{A}(\overline{v}(K))$.

Theorem (Gu.,Ka.)

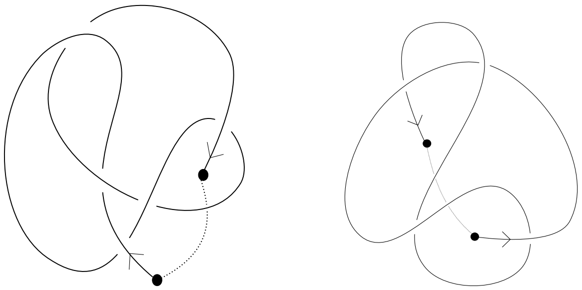
The height of a knotoid K in S^2 is greater than or equal to the Λ-degree of its arrow polynomial.

Idea of the Proof

- Closing a knotoid K virtually $\Leftrightarrow$ Closing the endpoints of the long segment component virtually.
- The Λ_i-variable of $\mathscr{A}[K]$ turns into the K_i-variable assigned to the circular component obtained.
- The Λ-degree of $\mathscr{A}[K] \leq$ # of the knot $\overline{v}(K)$, by the theorem of Dye and Kauffman.
- # of virtual crossings of $\overline{v}(K) \leq h(K)$, for any knotoid diagram K equivalent to K.
- The Λ-degree of $\mathscr{A}[K] \leq h(K)$
- The Λ-degree of $\mathscr{A}[K] \leq h(K)$

Which lower bound is better?

Back to the following two diagrams,

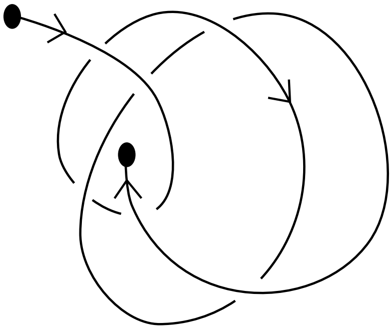


Both knotoids have trivial affine index polynomial. But,

- $\mathscr{A}[K_1] = -A^{-3} + A - 2A^5 + A^9 + A^{-9}\Lambda_1 - 2A^{-5}\Lambda_1 + 2A^{-1}\Lambda_1 - 2A^3\Lambda_1 + A^7\Lambda_1$. This tells the height of the knotoid is in fact 1.
- $\mathscr{A}[K_2] = \Lambda_1 + A^{-8}\Lambda_1 - 2A^{-4}\Lambda_1 - A^{-2}\Lambda_2 + A^2(1 + \Lambda_2)$. Therefore it is a height 2 knotoid.

A Question

What is the height of the following knotoid?



We find that $P_K(t) = 2t + 2t^{-1} - 4$ and $\mathscr{A}[K] = -A^{-5} + 2A^{-1} - A^3 - A^7 + 2A\Lambda_1 - 2A^5\Lambda_1$. So the polynomials both assure that the height is at least 1. But what is the height of the knotoid; 1 or 2?

Further Directions

- We want to know more about the height of knotoids and its relations with both the affine index polynomial and the arrow polynomial!!
  Does there exist an example for which the index polynomial is superior to the arrow polynomial in height determination?

- Determination of the kernel of the virtual closure map: *Is there a proper knotoid (a classical knotoid with nonzero height) whose virtual closure is the trivial knot?*
- Determination of the image of the virtual closure map: *How to determine if a given virtual knot is in the image of* $\overline{v}$ *or not? Is the virtual closure map surjective?*
- A generalization of the first question: *Is there a proper knotoid whose virtual closure is a classical knot* or *do proper knotoids always close (virtually) to a virtual knot of genus 1?*
- A generalization of the Jones polynomial conjecture: *The Jones polynomial for knotoids in* S^2 *detects the triviality of classical knotoids.*