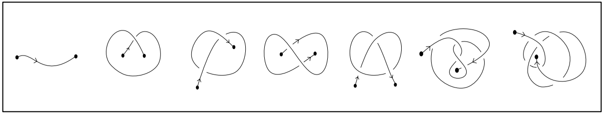

Knotoids and some invariants of them

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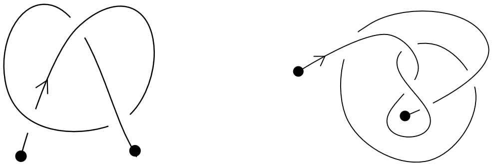
What is a knotoid diagram?



A *knotoid diagram* is an open-ended knot diagram in S^2 (or $\mathbb{R}^2$) with two endpoints that can lie in different regions of the diagram.

What is a knotoid diagram?

Definition (Turaev, 2012)

A *knotoid diagram* K in S^2 (or $\mathbb{R}^2$) is an immersion

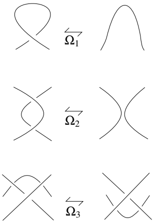
$$K : [0,1] \to S^2 \text{ or } \mathbb{R}^2 \text{ such that:}$$

1. each transversal double point is endowed with under/over-data, and we call them *crossings* of K,
2. the images of 0 and 1 are two disjoint points regarded as the *endpoints* of K. They are called the tail and the head of K, respectively.
3. K is oriented from the tail to the head.

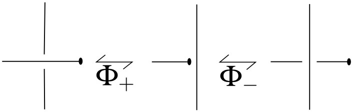
What is a knotoid?

Definition

A *knotoid* is an equivalence class of the knotoid diagrams up to the equivalence relation induced by the $\Omega_{i=1,2,3}$-moves plus the isotopy of S^2 (or of $\mathbb{R}^2$, respectively).



Equivalence moves



Forbidden knotoid moves

What is an invariant of a knotoid?

Definition

Let M denote a set of mathematical objects. An *invariant of knotoids* is a mapping

$$I : \{\text{Knotoid diagrams}\} \to M,$$

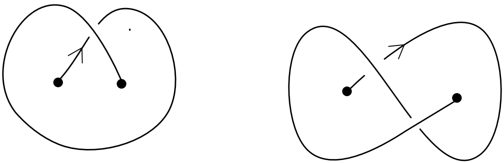
assigning the same object to equivalent knotoid diagrams.

A small note on knotoids in S^2 and in $\mathbb{R}^2$

There is a well-defined map,

$$\iota : \{\text{Knotoids in } \mathbb{R}^2\} \to \{\text{Knotoids in } S^2\}$$

induced by $\iota : \mathbb{R}^2 \hookrightarrow S^2$. This map is surjective, but not injective.



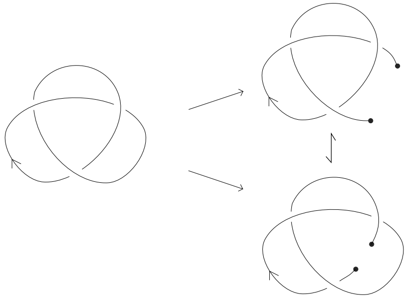
Nontrivial planar knotoids which are trivial in S^2

From classical knots to knotoids

- The theory of knotoids in S^2 is an extension of the classical knot theory. The map α,

$$\alpha\colon \text{ Classical knots} \to \text{Knotoids in } S^2$$

is induced by deleting an open arc which does not contain any crossings from an oriented classical knot diagram. The map α is injective.

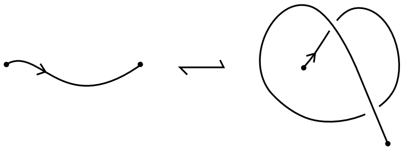


From classical knots to knotoids

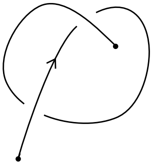
Definition

A knotoid in S^2 that is in the image of α, is called a *knot-type* knotoid. A knotoid that is not in the image of α is called a *proper* knotoid.

$$\{\text{Knotoids in } S^2\} = \{\text{Knot-type knotoids}\} \cup \{\text{Proper Knotoids}\}$$



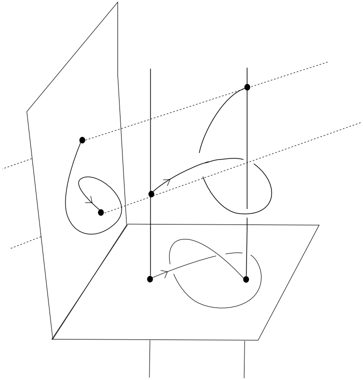
A knot-type knotoid



A proper knotoid

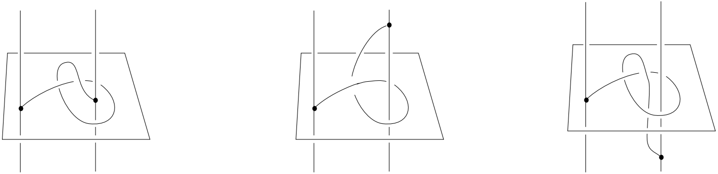
A geometric interpretation of knotoids in $\mathbb{R}^2$

- Any open-ended oriented space curve determines a knotoid in a plane: There are two lines passing through the endpoints and perpendicular to a plane where the curve has a generic projection. The projection of the curve along these lines on this plane gives a knotoid diagram.



A geometric interpretation

- Any knotoid determines an open-ended oriented curve in $\mathbb{R}^3$: Keep the endpoints attached on the two lines passing through the endpoints and perpendicular to the plane of a knotoid diagram and push the crossings up or down in the vertical direction accordingly to their over/under-data.



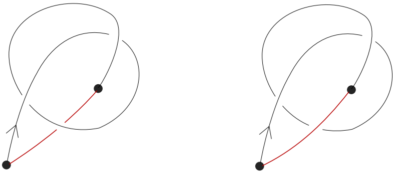
A geometric interpratation of knotoids in $\mathbb{R}^2$

Theorem (G. - Kauffman)

Two open-ended oriented curves embedded in $\mathbb{R}^3$ that are both generic to a given plane, are line isotopic (with respect to the lines determined by the endpoints of the curves and the plane) if and only if the projections of the curves to that plane are equivalent knotoid diagrams in the plane.

From knotoids to classical knots

- Knotoids give a new diagrammatic approach to knot theory: Any knotoid diagram *represents* a knot in $\mathbb{R}^3$ via the *over*- or *under-closure* and any classical knot can be represented by a knotoid diagram.



Two types of closures resulting in different classical knots

The underpass closure of a knotoid diagram induces a well-defined, surjective map,

$$\omega_{-}: \text{Knotoids} \rightarrow \text{Classical knots}$$

The height of a knotoid

Definition

The *height of a knotoid diagram* is the minimum number of crossings created during the underpass closure.
The *height of a knotoid* K is defined as the minimum of the heights, taken over all equivalent knotoid diagrams to K.

Theorem (Turaev)

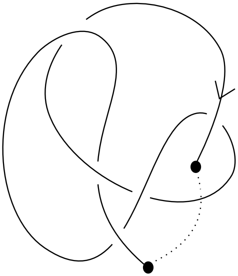
The height is a knotoid invariant.

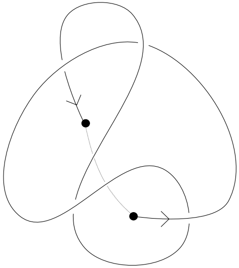
A knotoid has zero height iff it is a knot-type knotoid.

$$\equiv$$

A knotoid has non-zero height iff it is a proper knotoid.

How do we compute the height?





Question

Apparently, the first diagram has height 1 and the second one has height 2. Are there some equivalent diagrams to the diagrams above with less height?

Virtual knots

Virtual knot theory is a natural extension of the classical knot theory.

Definition (Kauffman)

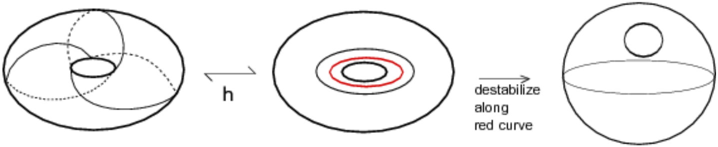
A *virtual knot k* is an embedding

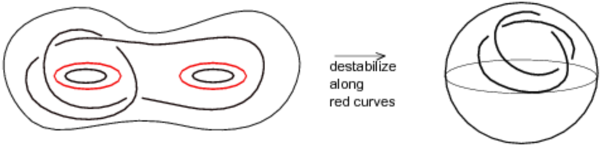
$$k : S^1 \to \Sigma_{g\geq 0} \times [0,1].$$

A *virtual knot diagram* is a generic projection of k in Σ_g with the data of under/over assigned to each self-crossings.

Definition

The virtual knot diagrams in two surfaces are said to be *stably-equivalent* if one is related to the other one by the three Reidemeister moves in the surfaces, orientation preserving homeomorphisms of the surfaces and addition/removal of 1- handles in the complements of the diagrams.





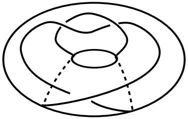
Combinatorial approach

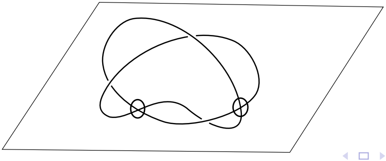
Definition (Kauffman)

A *virtual knot diagram* k is a generic immersion

$$k : S^1 \to \mathbb{R}^2 \text{ (or in } S^2\text{)},$$

with finitely many *classical crossings* and *virtual crossings*.

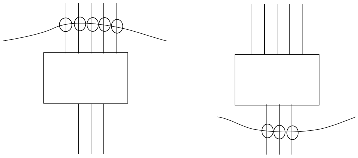




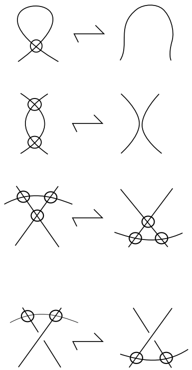
Combinatorial approach

Definition

Two virtual knot diagrams are *virtually equivalent* in S^2 or in $\mathbb{R}^2$ if there is a finite sequence of moves generated by the three Reidemeister moves and the *detour moves*.



The detour move



Theorem (Kauffman, Carter, Kamada, Saito)

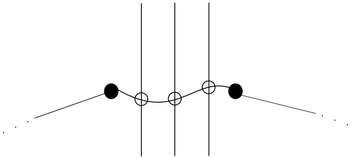
Two virtual knot diagrams are stably equivalent if and only if their corresponding diagrams in S^2 are virtually equivalent.

Definition

The *genus* of a virtual knot is the minimum genus among the surfaces that the knot has a diagram without any virtual crossings.

From knotoids to virtual knots

Every knotoid diagram in S^2 represents a virtual knot:
The endpoints of a knotoid diagram K are connected by an embedded arc in S^2 keeping the information of each intersection of the arc with the diagram as a virtual crossing.



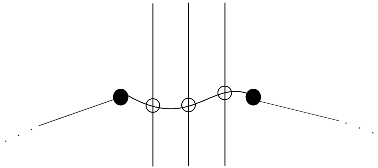
The resulting virtual knot diagram is called the *virtual closure* of K.

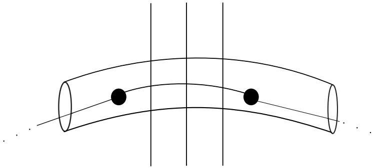
The virtual closure map

Definition

There is a well-defined map called the *virtual closure map*

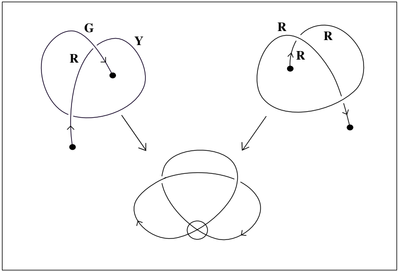
$$\bar{v} : \text{Knotoids in } S^2 \to \text{Virtual knots of genus} \leq 1 \ .$$





The virtual closure map

- The virtual closure map is not injective.



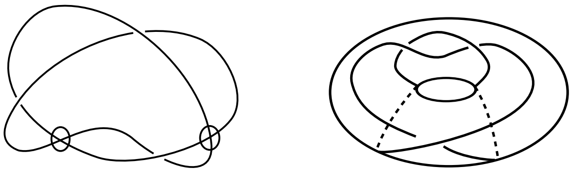
A pair of nonequivalent knotoids with the same virtual closure

The virtual closure map

Proposition (G. - Kauffman)

The virtual closure map is not surjective.

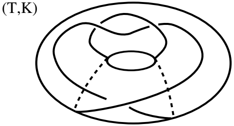
The following virtual knot is of genus 1 but it is not in the image of the virtual closure map:

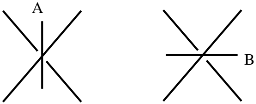


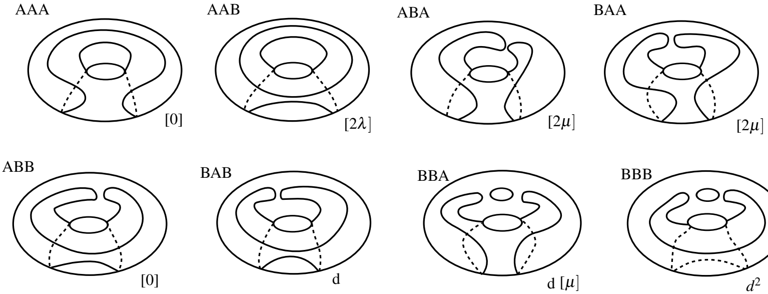
This can be shown by examining the surface-state curves of the diagram in torus.

Sketch of the proof

Observation 1: Non-trivial state curves of the following knot diagram are of the form $2[\lambda]$ and $2[\mu]$.







Proposition (G. - Kauffman)

If the state curves of a torus representation of a genus 1 *knot that is in the class of the closure of a knotoid, consist in simple closed curves of the form (up to some orientation)* $k[\lambda]$ *and* $m[\mu]$ *then* $|k| = |m| = 1$.

Conclusion

The virtual knot in question is not the virtual closure of a knotoid in S^2. Therefore the virtual closure map is not surjective.

Motivation

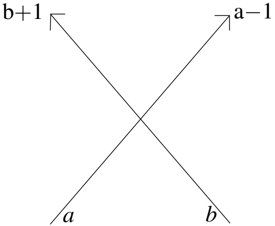
Having constructed a well-defined map, we can apply virtual knot invariants to knotoids!

Let Inv denote an invariant of virtual knots and K be a knotoid diagram. Define a knotoid invariant I by the following formula,

$$I(K) = Inv(\overline{v}(K)).$$

An affine index polynomial of knotoids

The invariant depends upon labeling the flat diagram corresponding to a knotoid diagram K with integers using following rule:



Integer labeling at a flat crossing

Note that the first and the last arc of the diagram can be always labeled by 0.

The affine index polynomial of a knotoid

Definition

Let c be a crossing of K, two number outcomes are derived, $w_+(c)$ and $w_-(c)$ which are called *positive* and *negative* weights of c:

$$w_+(c) = a - (b+1)$$
$$w_-(c) = b - (a-1)$$

where a and b are the labels for the left and the right incoming arcs at the corresponding flat crossing to c, respectively.
The *weight* of c is defined as

$$w_K(c) = \begin{cases} w_+(c) & \text{if the sign of } c \text{ is positive,} \\ w_-(c) & \text{if the sign of } c \text{ is negative} \end{cases}$$

The affine index polynomial of a knotoid

Definition

The *affine index polynomial* of a knotoid K is defined by the equation,

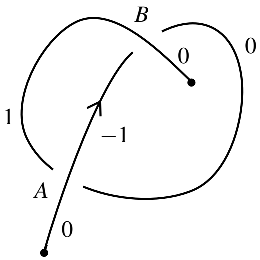
$$P_K(t) = \sum_c \text{sign}(c)(t^{w_K(c)} - 1)$$

where the sum is taken over all crossings of a diagram of K.

Theorem (G. – Kauffman)

The affine index polynomial is a knotoid invariant.

An example



		w_+	w_-
+	A	-1	1
+	B	1	-1

We find that
$P(K) = t + t^{-1} - 2.$

A comparison of the affine index polynomials

The affine index polynomial was defined as an invariant of virtual knots by L. Kauffman in 2012.

Theorem (Kauffman)

The affine index polynomial is trivial for all classical knots.

Facts

1. Knot-type knotoids have trivial affine index polynomial.
2. Proper knotoids may have nonzero affine index polynomial.
   $\Rightarrow$If a given knotoid diagram has nonzero affine index polynomial, then the knotoid diagram represents a proper knotoid.

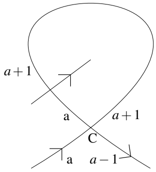
The affine index polynomial and the height

Theorem (G. – Kauffman)

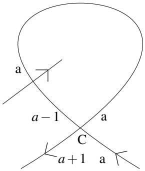
Let K be a knotoid diagram and m be the maximum degree of the affine index polynomial of K. Then the height of K, $h(K) \geq m$.

A sketch of the proof

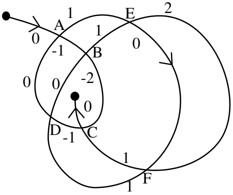
Observation 1: The algebraic intersection number of the loop at c with the strands intersecting the loop is equal to either $w_-(c)$ or $w_+(c)$.



Entering Value − Exit Value $= w_-(C)$

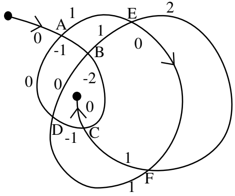


Entering Value − Exit Value $= w_+(C)$



	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1

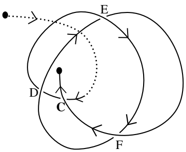
$$P_K(t) = t^2 + 2t + 2t^{-1} + t^{-2} - 6$$

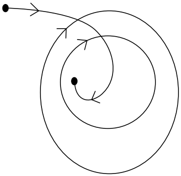


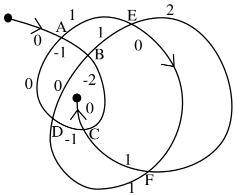
	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1

$$P_K(t) = t^2 + 2t + 2t^{-1} + t^{-2} - 6$$

We smooth twice-met crossings on the loop at a maximal weight crossing in the oriented way (crossing C for the diagram above):



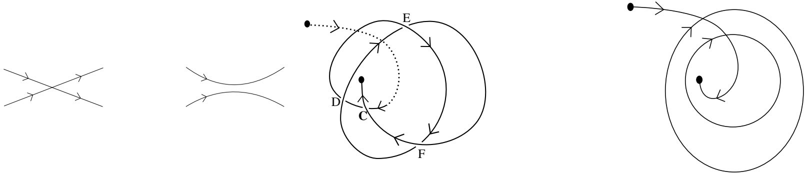




	w_+	w_-
A	-1	1
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1

$P_K(t) = t^2 + 2t + 2t^{-1} + t^{-2} - 6$

We smooth twice-met crossings on the loop at a maximal weight crossing in the oriented way (crossing C for the diagram above):



Observation 2: Let I_K be the algebraic intersection number of the long component with the Seifert circles. $|I_K| = w_K(C)$. Then the weight of C can be at most as the number of the Seifert circles.

Observation 3: The height of the diagram can be at least as the number of the Seifert circles by Jordan curve theorem.
Corollary:

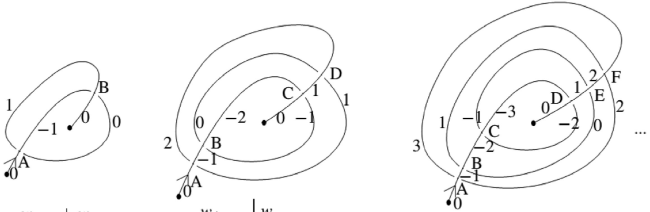
$$w(C) \leq h(K)$$

Since the maximum degree of the polynomial is invariant we have,

$$\text{max. degree of } \mathrm{P}_K(t) \leq \mathrm{h(K)}$$

for any equivalent diagram K.

Examples



	w_+	w_-
A	-1	$+1$
B	$+1$	-1

	w_+	w_-
A	-2	$+2$
B	-1	$+1$
C	$+2$	-2
D	$+1$	-1

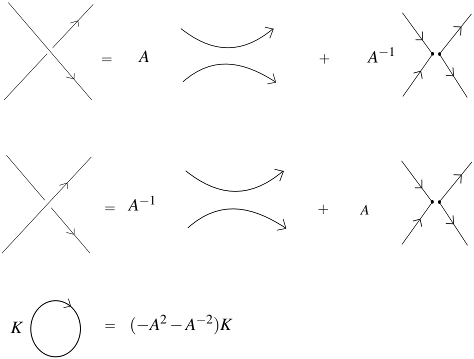
	w_+	w_-
A	-3	+3
B	-2	+2
C	-1	+1
D	+3	-3
E	+2	-2
F	+1	-1

- $P(K_1) = t + t^{-1} - 2.$
- $P(K_2) = t^2 + t + t^{-1} + t^{-2} - 4.$
- $P(K_3) = t^3 + t^2 + t + t^{-1} + t^{-2} + t^{-3} - 6.$

These give examples of height n for any natural number n.

The arrow polynomial

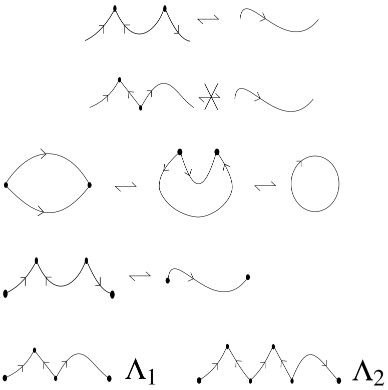
The construction of the arrow polynomial of knotoids is based on the *oriented state expansion* of the bracket polynomial of knotoids.



Oriented state expansion

Reduction rules for the arrow polynomial

To reduce the number of cusps in a state component we have the following rules:



The arrow polynomial

Definition

We define the *arrow polynomial* of a knotoid diagram K as,

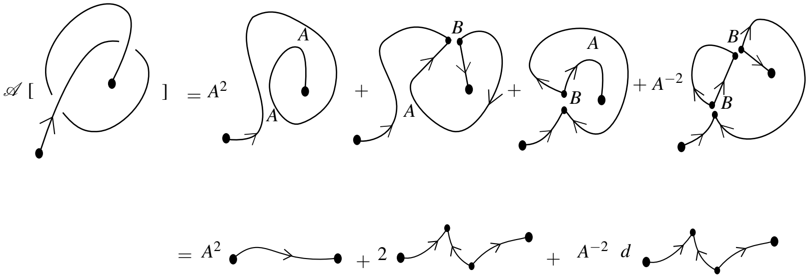
$$\mathscr{A}[K] = \sum_S < K|S > (-A^2 - A^{-2})^{\|S\|-1} \Lambda_i$$

where the sum runs over the oriented bracket states, $< K|S >$ is the usual vertex weights of the bracket polynomial, $\|S\|$ is the number of components of the state S and Λ_i is the variable associated to the long segment component of S with irreducible zig-zags.

Theorem (G.–Kauffman)

The normalization of the arrow polynomial (multiplication by $(-A^3)^{-\operatorname{wr}(K)}$*) is a knotoid invariant.*

An example



$$= A^2 + (1 - A^4)\Lambda_1$$

The arrow polynomial and the height

Definition

The Λ-degree of a summand of the arrow polynomial of a knotoid which is in the form, $A^m\Lambda_i$ is equal to i. The Λ-*degree* of the arrow polynomial of a knotoid is defined to be the maximum Λ- degree among the Λ-degrees of all the summands of the polynomial.

The arrow polynomial and the height

Definition

The Λ-degree of a summand of the arrow polynomial of a knotoid which is in the form, $A^m \Lambda_i$ is equal to i. The Λ-*degree* of the arrow polynomial of a knotoid is defined to be the maximum Λ- degree among the Λ-degrees of all the summands of the polynomial.

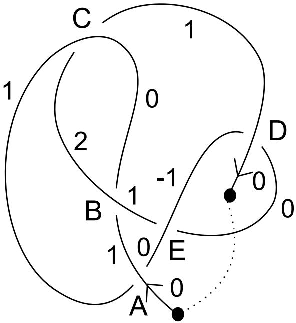
Theorem (G. – Kauffman)

The height of a knotoid K in S^2 is greater than or equal to the Λ-degree of its arrow polynomial.

Idea of the proof

- Closing a knotoid diagram K virtually corresponds to closing the endpoints of the long segment components of the states of K virtually.
- The Λ_i-variables of $\mathscr{A}[K]$ turns into the K_i-variables assigned to the circular components obtained.
- A theorem of Kauffman and Dye tells that the K-degree of the arrow polynomial is a lower bound for the virtual crossing number.
- The Λ-degree of $\mathscr{A}[K] \leq$ # of the virtual crossings of the knot $\overline{v}(K)$.
- # of virtual crossings of the knot $\overline{v}(K) \leq h(K)$, for any knotoid diagram equivalent to K.
- The Λ-degree of $\mathscr{A}[K] \leq h(K)$.

Which lower bound is better?



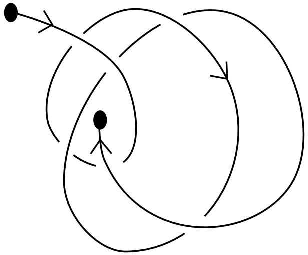
A	0	0
B	-1	1
C	1	-1
D	1	-1
E	-1	1

- $P(K) = (t^0 - 1) - (t-1) - (t^{-1} - 1) + (t-1) + (t^{-1} - 1) = 0$
- $\mathscr{A}[K_1] = -A^{-3} + A - 2A^5 + A^9 + A^{-9}\Lambda_1 - 2A^{-5}\Lambda_1 + 2A^{-1}\Lambda_1 - 2A^3\Lambda_1 + A^7\Lambda_1.$

Conclusion: The height of the knotoid is 1.

A question

What is the height of the following knotoid?



$P_K(t) = 2t + 2t^{-1} - 4$ and
$\mathscr{A}[K] = -A^{-5} + 2A^{-1} - A^3 - A^7 + 2A\Lambda_1 - 2A^5\Lambda_1$. Both polynomials assure that the height is at least 1.

What is the height of the knotoid; 1 or 2?

A conjecture on the height

Conjecture (Turaev)

Minimal crossing diagrams of knot-type knotoids have zero height.

A recipe to prove the conjecture

Main Ingredient:

Theorem (Manturov)

Let κ be an isotopy class of a classical knot. Then the minimal number of classical crossings for virtual diagrams of κ is realized on classical diagrams (genus 0- diagrams) up to detour moves.

A recipe to prove the conjecture

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Let κ be an isotopy class of a classical knot. Then the minimal number of classical crossings for virtual diagrams of κ is realized on classical diagrams (genus 0- diagrams) up to detour moves.

Sketch of the proof:

1. Let k be a knot-type knotoid then $\overline{v}(k)$ is a classical knot.
2. Assume there exists a minimal crossing knotoid diagram K with nonzero height. This forces the virtual knot diagram $\overline{v}(K)$ to be a minimal diagram for $\overline{v}(k)$.
3. The underlying genus of $\overline{v}(K)$ is 1.
4. This contradicts with the above theorem.

Corollary

Crossing number of a knot-type knotoid is equal to the crossing number of the knot that is the underpass closure of the knotoid.

Further Questions & Directions

- We want to know more about the height of knotoids and its relations with both the affine index polynomial and the arrow polynomial.
  Does there exist an example for which the index polynomial is superior to the arrow polynomial in height determination?

- How to determine if a virtual knot of genus 1 is in the image of the virtual closure map?
  We have the following conjectures on the characterization of the kernel and the image of the virtual closure map.

Conjecture

The virtual closure of a proper knotoid is always a genus 1 virtual knot.

Conjecture

There is no proper knotoid whose virtual closure is the trivial knot.

Thank you for your attention!